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STABILITY OF

LONGITUDINALLY STIFFENED CYLINDRICAL SHELLS

SUBJECTED TO AXIAL COMPRESSION

A THESIS

Presented to

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STABILITY OF

LONGITUDINALLY STIFFENED CYLINDRICAL SHELLS

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Approved:

Chairman

Date approved by Chairman: July 31, 1970

To

my Parents, my Parents in Law

and my Wife

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LIST OF SYMBOLS

All symbols are defined in the text where they first appear and the major symbols are listed below:

a	radius of the shell
C	stringer torsional rigidity ($=G_s J$)
C_w	stringer warping constant
C_1	stringer warping rigidity ($=E_s C_w$)
d	distant between two adjacent stringers divided by shell radius
D	bending stiffness $\left(= \frac{Eh^3}{12(1-\nu^2)}\right)$
E	Young's modulus of the skin
E_s	Young's modulus of stringers
F	stress function
h	thickness of the shell
h_y, h_z	coordinates of skin and stringer contacting point
I_o	polar moment of inertia of stringer cross-sectional area about its shear center
I_y, I_z, I_{yz}	moments and product of inertia of the stringer cross-sectional area about its centroidal axes
ℓ	length of the shell
L	length of the shell divided by shell radius
M_x, M_{xy}, M_y	stress couples in the shell
N_x, N_{xy}, N_y	stress resultants in the shell
p	average axial compressive stress in the skin

q_x, q_y, q — longitudinal, circumferential and normal surface loading components on the shell

R_{1m}, R_{1mn} — half of the number of the real roots of the characteristic equations

R_{2m}, R_{2mn} — half of the number of the pure imaginary roots of the characteristic equations

R_{3m}, R_{3mn} — one fourth of the number of the complex roots of the characteristic equations

u, v, w — longitudinal, circumferential and normal displacement components of the shell

u_N, v_N, w_N — stringer displacement components along the line of contact with the skin

u_o, v_o, w_o — pre-buckling displacement components of the shell

u_s, v_s, w_s — longitudinal, circumferential and normal displacement components of stringers

V_x, V_y — effective transverse shear stress resultants in the shell

y_o, z_o — coordinates of shear center of stringers

Z — curvature parameter $\left(= \frac{\ell^2}{ah} (1-\nu^2)^{\frac{1}{2}}\right)$

$\alpha_{mj}, \alpha_{mnj}$ — real roots of the characteristic equation

β_{mj}, β_{mnj} — pure imaginary roots of the characteristic equation

$\gamma_{mj}, \gamma_{mnj}, \delta_{mj}, \delta_{mnj}$ — real and imaginary parts of complex roots of the characteristic equation

ξ, η, ζ — dimensionless cylindrical coordinates (cylindrical coordinates divided by shell radius)

σ	dimensionless average axial compressive stress in the skin $\left(= \left(\frac{1}{2}\, p/E\right)^{\frac{1}{2}}\right)$
ρ	$= \sqrt{12(1-\nu^2)}\left(\frac{a}{h}\right)$
ν	Poisson's ratio
ϕ_s	angle of twist of the stringer cross-section

SUMMARY

This dissertation is concerned with a theoretical investigation of the stability characteristic of a thin cylindrical shell stiffened longitudinally by stringers with classical simply supported boundary conditions subjected to axial compression. The usual assumption that the stringers are closely spaced is not made and hence the stiffened shell is not treated as an orthotropic continuum but rather as an isotropic cylinder interacting with a set of discrete thin walled beams.

The analysis is developed from a fundamental viewpoint of a system of coupled differential equations suitable for the investigation of the stability of longitudinally stiffened cylindrical shells as an eigenvalue problem. Large deformation shell theory using Donnell's approximations and linear theory for stringers are used in the general formulation of the problem. Torsion, flexure and longitudinal contraction of the stringers are included in this study and the eccentricity effects of the stringers are automatically accounted for.

The analysis based upon the linearized Donnell-type equation is made first. Sinusoidal functions for radial displacement are assumed in the longitudinal direction and the exact solutions in the circumferential direction are solved for shells. By matching the continuity conditions for the deformations and the compatibility conditions for stresses along the stringer lines, a linear eigenvalue problem for determination of the critical load of the composite structure is yielded.

The analysis is also made for the shell stiffened by equally spaced symmetric stringers by taking into account the effects of the pre-buckling deformation. By assuming the form of the double Fourier series, the pre-buckling deformation is obtained by an iterative method first. The linear partial differential equation with non-constant coefficients governing the radial buckling increment is then solved by the Galerkin method.

Numerical examples are presented in Chapter V for illustrative purposes and some interesting phenomena are found.

CHAPTER I

INTRODUCTION

The demand for designs of efficient structures in modern vehicles leads to the requirement that monocoque shells be avoided[1,2]. The construction of stiffened shells is one of several methods used to accomplish this goal. However, the buckling behavior of such stiffened shells differs considerably from that of thin monocoque shells. A great deal of investigation on the buckling problems of stiffened shells has been previously carried out.

The first theoretical investigation on the buckling of an infinitely long uniform cylinder subjected to hydrostatic pressure was carried out by Bress[3] in 1859. Based upon an energy criterion, in 1888 Bryan[4] derived expressions similar to those of Bress. During 1913-1915, Southwell[5] published a series of three papers concerning the elastic stability of cylindrical shells. In the first of these papers he discussed the effect of circumferential reinforcing rings on the buckling of a shell subjected to hydrostatic pressure.

In the 1930's, theoretical investigations on the buckling of the stiffened cylindrical shells were performed by Flügge[6], Dschou[7], Taylor[8], Timoshenko[9], Nissen[10], and Rydner[11]. Flügge developed the fundamental theory for the buckling of an orthotropic cylindrical shell as an extension of three equilibrium equations which he had derived for an isotropic cylindrical shell. Dschou and Timoshenko solved these equations for a stiffened circular cylindrical shell as an equivalent orthotropic shell subjected to an axial load. In their analyses, Dschou neglected the torsional stiffness but Timoshenko included the torsional stiffness. Taylor derived an eighth order partial differential equation for axially loaded orthotropic cylindrical shells utilizing the same approach as Donnell's[12] approximate theory which had been developed earlier for isotropic circular cylindrical shells. Nissen's work was concerned with the correlation of experimental results with the theoretical work of Dschou. Rydner modified and simplified the works of Taylor and Timoshenko, and also introduced the additional reduction factor to take into account the discrepancies found between theory and experiment for the unstiffened cylinder.

Later, Hoff[13] employed an energy method by distributing the rigidities of longitudinal stiffeners but considering the rings as local elastic supports. Wang[14] used an energy method to study the buckling of stiffened shell under axial compression. Salerno and Levine[15], Kendricks[16], Nash[17] and others solved the problem of ring-reinforced cylindrical shells subjected to hydrostatic pressure by an energy method. Stein, Sanders and Crate[18] investigated analytically the behavior of a ring stiffened cylindrical shell under torsion. Anderson and Card[19] studied the thermal buckling problem of ring-stiffened cylindrical shells. Wood[20] and McKenzie[21] studied pressurized stiffened cylindrical shells subjected to axial loads. Hoff, Boley and Klein[22] used an energy method, similar to Salerno's, to analytically study a stiffened shell with cutouts, they also presented experimental verification of their results. Dickson and Brolliar[23] investigated the general instability of eccentrically stiffened cylindrical shells subjected to a combination of axial compression and lateral pressure where the local buckling of skin was examined prior to the analysis of the general instability. Simitses[24] used the Galerkin method to study the buckling of stiffened cylindrical shells which had been subjected to combined torsion and hydrostatic pressure.

Some experimental work was carried out by Tokugawa[25], Galletly, Slankard and Wenk[26], Harris, Suer and Skene[27], Peterson, Whitley and Deaton[28], Shang, Marulic and Sturn[29], Milligan, Gerard, Lakshmikantham and Becker[30], Singer[31], and Tsao and Ching[32].

As early as 1947, Van der Neut[33] demonstrated the importance of the eccentricity of stiffening elements in determining the buckling stress of a stiffened cylindrical shell. Singer, Baruch and Harari[34, 35,36], Hepdgepeth and Hall[2], Block, Card and Mikulas[37], McElman, Mikulas and Stein[38], and Simitses[39] studied the eccentricity effects of stiffeners. Houghton and Chan[40] indicated that some data obtained according to tests made in the College of Aeronautics, Cranfield, confirmed the importance of the eccentricity effect. Further experimental evidence can be found in tests carried out by Card[41], DeLuzio, Stuhlman and Almroth[42], and by Garkisch, Geier and Seggelke[43]. The physical explanation of the effect of eccentricity of rings on the instability of the stiffened cylindrical shell under the hydrostatic pressure has been given in [36] and that of stringers under an axial compressive force in [35]. A partial physical explanation of the effect in stringer stiffened shells has also been given by Thielemann and Esslinger[44]. It has been indicated in [35] that the behavior of the eccentricity effect depends very strongly on the geometry of the shell while the geometry of the stiffeners only influences its magnitude. It has been shown that outside stringers yield higher buckling loads than inside ones for all practical long longitudinally stiffened cylindrical shells. In fact, the calculations as well as the tests show that the externally stiffened cylindrical shell can carry more than three times the load sustained by its internally stiffened counterpart. However, in a few instances, the magnitude of the buckling load for a cylinder with inside stringers is slightly larger than that of outside ones for the short stiffened cylinder.

Brush[45] used the finite difference method to investigate the imperfection sensitivity of the ring and stringer stiffened cylindrical shell subjected to an axial compression by taking into account the eccentricity and torsional rigidity of stiffeners. Hutchinson and Amazigo[46] gave a quantitative study of the imperfection sensitivity of an eccentrically stiffened cylindrical shell within the framework of Koiter's general theory of postbuckling behavior[47-49] for both axial and ring stiffened cylinders under an axial compression and hydrostate pressure. It has been concluded in [46] that

> in some instances, in particular in the case of axially stiffened cylinders under axial load, the sensitivity to imperfection, as well as the classical buckling load, appears to be strongly dependent on whether the stringers are attached to the outside or inside of cylinder, and under certain conditions stiffening can significantly reduce or perhaps completely eliminate imperfection-sensitivity whereas in other cases it may play a much smaller role in lowering the sensitivity.

Jones[50] investigated the buckling of circular cylindrical shell with multiple orthotropic layers and eccentric stiffeners.

All of the investigations cited above have treated the stiffened shells by smearing the stiffness of the skin and stiffeners together and considering the structure as an equivalent homogeneous orthotropic cylindrical shell. Such an assumption made for idealizing a stiffened shell is reasonable if the stiffeners are closely spaced. Otherwise it will be desirable to treat the skin and stiffeners as distinct structural elements.

Stein, Sanders and Crate[18], Moe[51], Block[52], Van der Neut[53] MacNeal, Winemiller and Bailie[54], Ross[55], and Singer and Haftka[56] have treated ring-reinforced cylindrical shells by considering circumferential rings as discrete elements. Stein, Sanders and Crate investigated the buckling problem of the stiffened cylindrical shell under torsion. Moe, Ross, MacNeal, Winemiller, and Bailie studied the buckling of stiffened shells under lateral pressure. Singer and Haftka employed the Dirac delta function, took into account the eccentricity effect of ring stiffeners, and concluded that the discreteness effect of rings depends very strongly on the geometry of the shell and the eccentricity of the rings.

So far, no analytical work on the buckling of a longitudinally stiffened shell, treating the stiffeners as discrete components, can be found in the published literature. Limited work on the problem of stiffened shells, e.g., Wang[57] made a stress analysis, and Egle and Sewall[58], Rinehart[59] and McDonald[60] studied the free vibration of stiffened cylindrical shell, has been done considering the stiffeners as discrete components.*

This analysis is developed from a fundamental viewpoint for a system of coupled differential equations suitable for the investigation of the stability of longitudinally stiffened cylindrical shells as an eigenvalue problem. Large deformation shell theory using Donnell's approximations and linear theory for stringers are used in the general formation of the problem. Torsion, flexure, and longitudinal contraction of stringers are all included in this analysis and the eccentricity effects of the stringers are automatically accounted for. Classical simply supported cylindrical shells are considered in the study. An analysis based on the linearized Donnell-type equation is first made. An analysis is also presented which takes into account the effect of the pre-buckling deformation.

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* A recent communication with Dr. Singer reveals that the investigations on the buckling of stringer-stiffened cylindrical shells by considering the stiffeners as linear discontinuities represented by the Dirac delta function are in progress, but the approach differs from this analysis.

CHAPTER II

GENERAL FORMULATION

Each stringer and each panel between every two adjacent stringers is taken as a separate structural component. Each panel is governed by an eighth order partial differential equation of Donnell type. Each stringer is governed by a set of four ordinary differential equations governing the linear displacement components of the shear center and the angle of twist of the cross-section according to the linear theory of thin walled beam of open cross-section. The interactions between the skin and the stringers are coupled through the continuity conditions for deformations and equilibrium conditions for stresses. The analysis is applicable to any number of stringers with any kind of cross-sectional shape.

Governing Differential Equations for Skin

By the following assumptions which were first advanced by Donnell[12], one can obtain the Donnell-type equations which are completely uncoupled such that one of the equations involves the radial displacement component, w, as the only dependent variable. Each of the remaining equations relates one of the other displacement components to w. The assumptions are:

(1) The transverse shearing force, Q_y, makes a negligible contribution to the equilibrium of the forces in the circumferential direc-

tion, i.e., the term $\frac{Q_y}{a}$ is neglected in the equation of equilibrium;

(2) The changes of curvature and twist are negligibly affected by the "stretching" displacement, v; i.e., the changes in curvature and twist of the thin cylindrical shell can be expressed as

$$K_x = \frac{\partial^2 w}{\partial x^2} \quad , \quad K_y = \frac{\partial^2 w}{\partial y^2} \quad , \quad K_{xy} = 2\frac{\partial^2 w}{\partial x \partial y} \; . \tag{2-1}$$

It is noted that these two assumptions are not independent, as can be shown by a consistent energy formulation of the differential equations.

The Donnell-type equations of large deflections of the thin cylindrical shell are:

$$D\nabla^8 W + \frac{Eh}{a^2}\frac{\partial^4 w}{\partial x^4} - \nabla^4\Big[\frac{\partial}{\partial x}\Big(N_x\frac{\partial W}{\partial x}\Big) + \frac{\partial}{\partial x}\Big(N_{xy}\frac{\partial W}{\partial y}\Big) + \frac{\partial}{\partial y}\Big(N_{xy}\frac{\partial W}{\partial x}\Big) \tag{2-2}$$

$$+ \frac{\partial}{\partial y}\Big(N_y\frac{\partial w}{\partial y}\Big)\Big] = \nabla^4 q - \frac{1}{a}\Big[\nu\frac{\partial^3 q_x}{\partial x^3} - \frac{\partial^3 q_x}{\partial x \partial y^2} + \frac{\partial^3 q_y}{\partial y^3} + (2+\nu)\frac{\partial^3 q_y}{\partial x^2 \partial y}\Big]$$

$$+ \frac{Eh}{a}\frac{\partial^2}{\partial x^2}\Big[\Big(\frac{\partial^2 w}{\partial x \partial y}\Big)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}\Big] ,$$

$$\nabla^4 u = -\frac{1}{a}\frac{\partial^3 w}{\partial x \partial y^2} + \frac{\nu}{a}\frac{\partial^3 w}{\partial x^3} - \frac{(1-\nu^2)}{Eh}\Big[\frac{\partial^2 q_x}{\partial x^2} + \frac{2}{1-\nu}\frac{\partial^2 q_x}{\partial y^2} - \frac{1+\nu}{1-\nu}\frac{\partial^2 q_y}{\partial x \partial y}\Big] , \tag{2-3}$$

$$\nabla^4 v = \frac{2+\nu}{a}\frac{\partial^3 w}{\partial x^2 \partial y} + \frac{1}{a}\frac{\partial^3 w}{\partial y^3} - \frac{(1-\nu^2)}{Eh}\Big[\frac{\partial^2 q_y}{\partial y^2} + \frac{2}{1-\nu}\frac{\partial^2 q_y}{\partial x^2} - \frac{1+\nu}{1-\nu}\frac{\partial^2 q_x}{\partial x \partial y}\Big] , \tag{2-4}$$

where D is the bending stiffness, E is Young's modulus of the skin, h is the skin thickness, a is the radius of the cylinder, q_x, q_y and q are loading components in x, y and normal directions, and ν is Poisson's ratio. The stress resultants (N_x, N_y, N_{xy}), the effective transverse shearing stress resultants (V_x, V_y), and stress couples (M_x, M_y, M_{xy}) can be expressed in terms of the displacement components (u, v, w) as follows:

$$N_x = \frac{Eh}{1-\nu^2}\left(\frac{\partial u}{\partial x} + \nu\frac{\partial v}{\partial y} - \nu\frac{w}{a}\right), \tag{2-5}$$

$$N_y = \frac{Eh}{1-\nu^2}\left(\frac{\partial v}{\partial y} - \frac{w}{a} + \nu\frac{\partial u}{\partial x}\right), \tag{2-6}$$

$$N_{xy} = \frac{Eh}{2(1+\nu)}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right), \tag{2-7}$$

$$V_x = -D\left(\frac{\partial^3 w}{\partial x^3} + (2-\nu)\frac{\partial^3 w}{\partial x\partial y^2}\right), \tag{2-8}$$

$$V_y = -D\left(\frac{\partial^3 w}{\partial y^3} + (2-\nu)\frac{\partial^3 w}{\partial x^2\partial y}\right), \tag{2-9}$$

$$M_x = -D\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right), \tag{2-10}$$

$$M_y = -D\left(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right), \tag{2-11}$$

$$M_{xy} = -D(1-\nu)\frac{\partial^2 w}{\partial x\partial y}, \tag{2-12}$$

where x, y and z are cylindrical coordinates as shown in Figure 1; w is taken positive as deflected inward, and the other sign conventions are shown in Figure 2.

The derivations of the above expressions are given in Appendix A. It is well known that the eighth order partial differential equation (Eq.(2-2)) may be represented by two fourth order partial differential equations which are presented in Appendix A. The accuracy of Donnell's equation has been discussed by several investigators. For stiffened cylindrical shells, Rinehart[59] concluded that Donnell's equation leads to a reliable result in the case of free vibration.

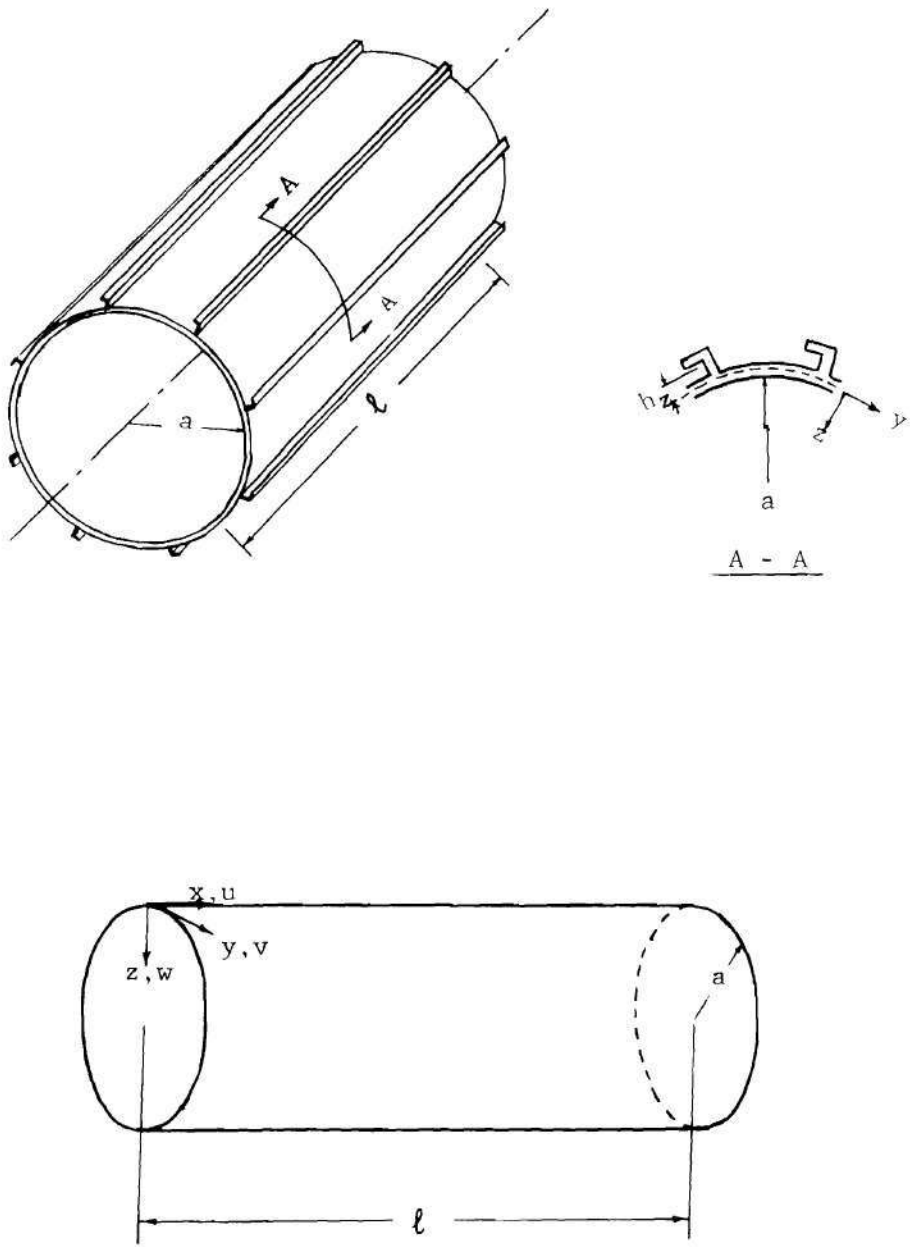


Figure 1. Geometry and Coordinate System

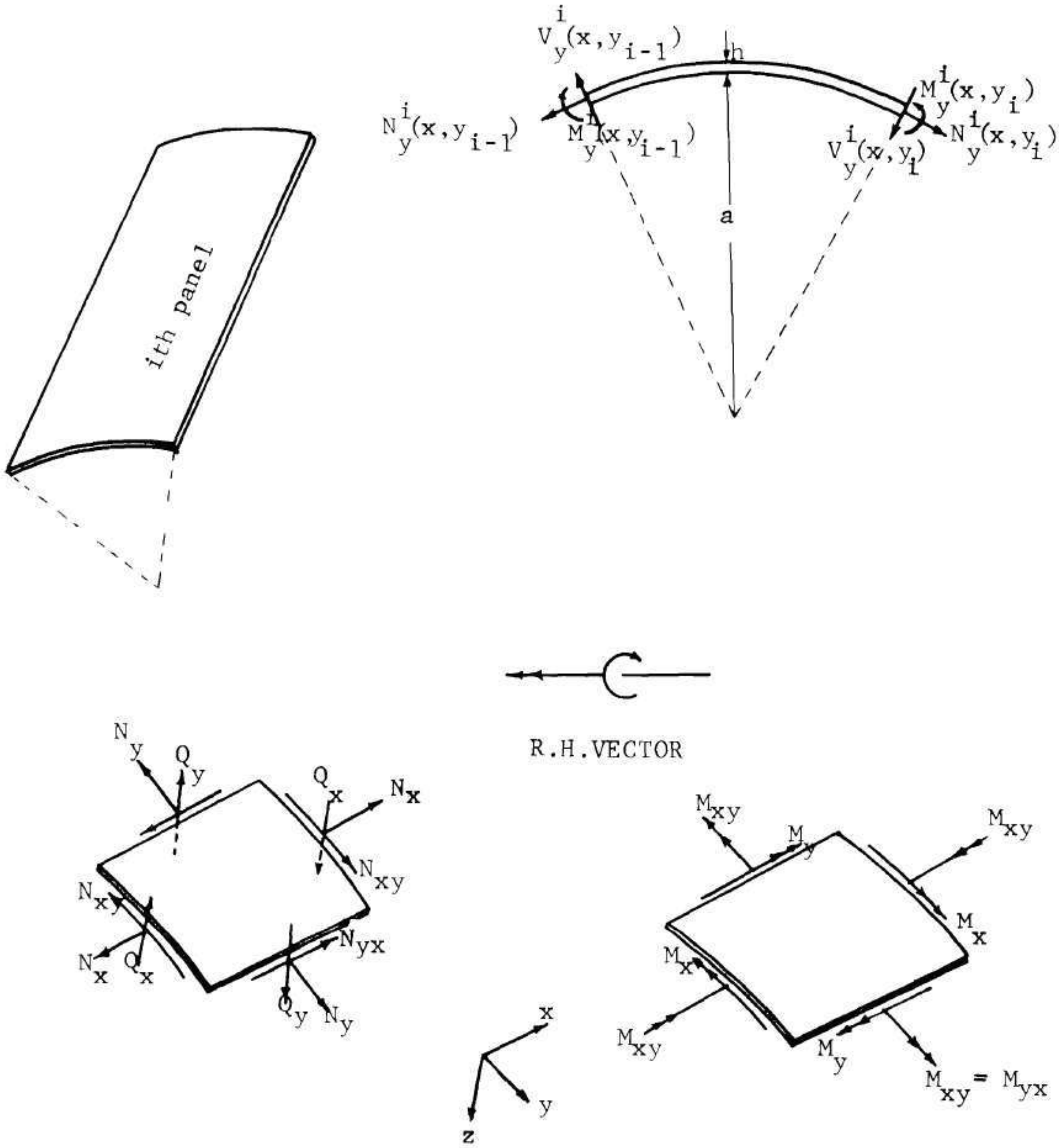


Figure 2. Sign Convention

Governing Differential Equations for Stringers

The stringers are considered as thin walled beams with open sections which are elastically attached to the skin throughout the length. The following assumptions are considered and the linear theory of open section thin walled beam will be used throughout this dissertation:

(1) The section contour is undeformable; i.e., the cross-sectional shape is preserved.

(2) Effect of shearing deformation in the middle surface of the beam is neglected.

For the linear displacement components, u_s, v_s and w_s, of the shear center, and the angle of twist, ϕ_s, of the cross-section of the stringers, the following four governing differential equations are derived in Appendix B with the nonlinear terms neglected:

$$E_s I_y \frac{d^4 w_s}{dx^4} + E_s I_{yz} \frac{d^4 v_s}{dx^4} + N_0 \frac{d^2 w_s}{dx^2} - y_0 N_0 \frac{d^2 \phi_s}{dx^2} = V_{ys} + h_z \frac{dN_{xys}}{dx} , \tag{2-13}$$

$$E_s I_z \frac{d^4 v_s}{dx^4} + E_s I_{yz} \frac{d^4 w_s}{dx^4} + N_0 \frac{d^2 v_s}{dx^2} + z_0 N_0 \frac{d^2 \phi_s}{dx^2} = N_{ys} + h_y \frac{dN_{xys}}{dx} , \tag{2-14}$$

$$-C_1 \frac{d^4 \phi_s}{dx^4} + \left(C - \frac{I_0}{A_s} N_0 \right) \frac{d^2 \phi_s}{dx^2} + N_0 y_0 \frac{d^2 w_s}{dx^2} - N_0 z_0 \frac{d^2 v_s}{dx^2} \tag{2-15}$$

$$= M_{ys} + (y_0 - h_y) V_{ys} - (z_0 - h_z) N_{ys} + \omega_{(s_n)} \frac{dN_{xys}}{dx} ,$$

and

$$-E_s A_s \left(\frac{d^2 u_s}{dx^2} + z_0 \frac{d^3 w_s}{dx^3} + y_0 \frac{d^3 v_s}{dx^3} \right) = N_{xys} , \tag{2-16}$$

where E_s is the Young's modulus of the stringer; I_y and I_z are centroidal moments of inertia with I_{yz}, the product of inertia of the stringer; C_1 is warping rigidity of the stringer; C is torsional rigidity of the stringer; A_s is cross-sectional area of the stringer; I_o is the polar moment of inertia about the shear center of the stringer; (y_o, z_o) are the coordinates of the shear center of the stringer; (h_y, h_z) are the coordinates of the contact point of the stringer and skin; V_{ys}, N_{ys}, N_{xys} and M_{ys} are interacting loads acting along the intersection lines of the stringer and the shell, as indicated in Figures 3 and 9. $\omega_{(s_N)}$ is the sectorial area evaluated at the skin and stringer contacting point N (see Appendix B).

Displacement Continuity and Stress Compatibility Conditions

The continuity conditions require that the displacements along the edges of the adjacent panels must be the same and equal to the corresponding displacements of the intersection line of the stringers and the shell. The slope of circumferential direction along the edges of the adjacent panels must be equal to the angle of the twist of the cross-section of the stringer. Hence,

$$w^i_{(x,y_i)} = w^{i-1}_{(x,y_i)} = w^i_N(x) \quad , \tag{2-17}$$

$$u^i_{(x,y_i)} = u^{i-1}_{(x,y_i)} = u^i_N(x) \quad , \tag{2-18}$$

$$v^i_{(x,y_i)} = v^{i-1}_{(x,y_i)} = v^i_N(x) \quad , \tag{2-19}$$

$$\left.\frac{\partial w^i}{\partial y}\right|_{y=y_i} = \left.\frac{\partial w^{i-1}}{\partial y}\right|_{y=y_i} = \phi^i_s(x) \quad , \tag{2-20}$$

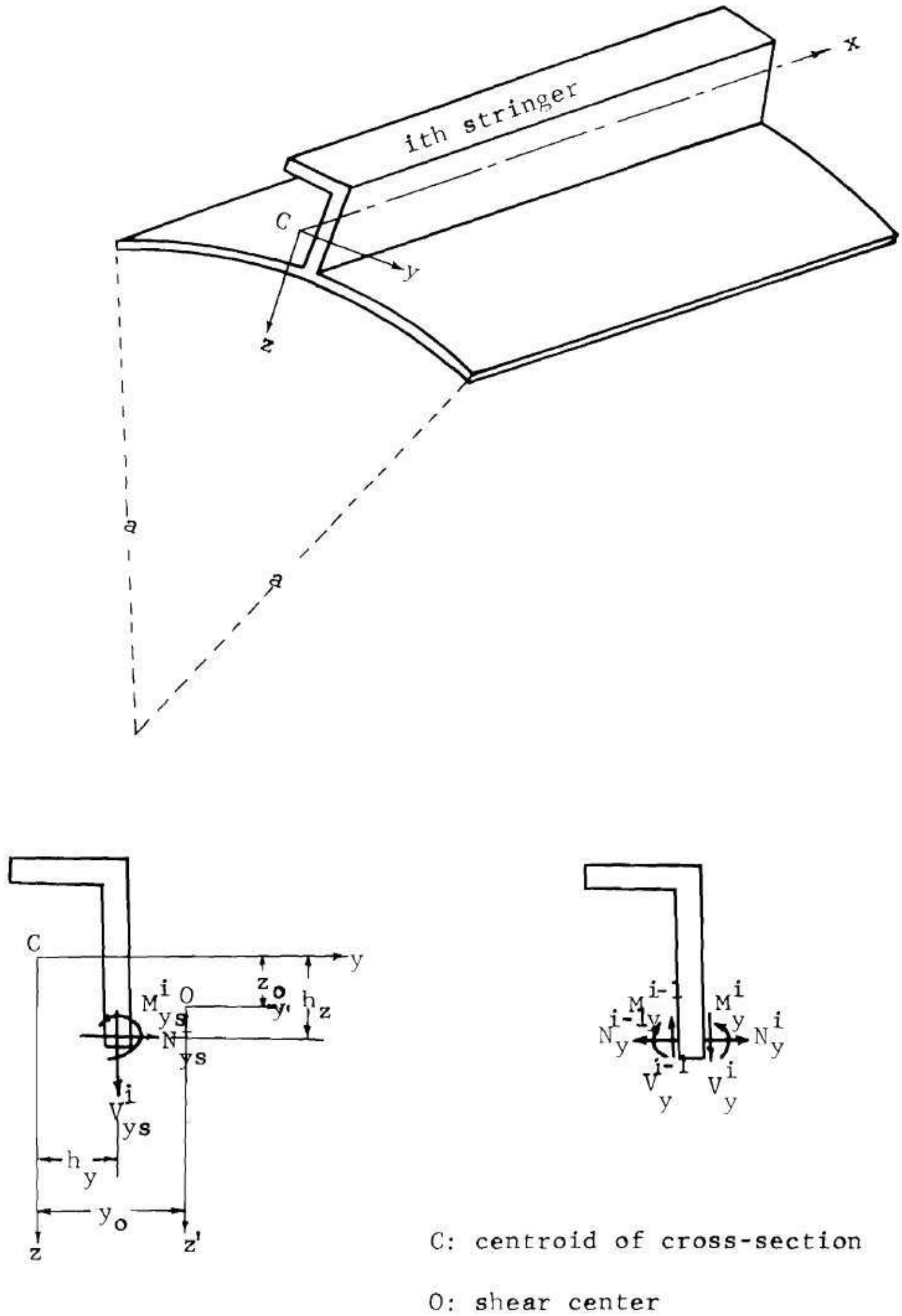


Figure 3. Stringer Coordinate System and Notation

where the superscript "i" implies the i^{th} panel or the i^{th} stringer.

The governing differential equations of the stringers are referenced to the linear displacement of the shear center and the angle of twist of the cross-section. The linear displacement components, u^i_N, v^i_N and w^i_N, at the intersection line with the skin can be expressed in terms of the linear displacement components, u^i_s, v^i_s and w^i_s, of the shear center as:

$$w^i_N = w^i_s - (y^i_o - h^i_y)\phi^i_s \quad \text{or} \quad w^i_s = w^i_N + (y^i_o - h^i_y)\phi^i_s \; ; \tag{2-21}$$

$$v^i_N = v^i_s + (z^i_o - h^i_z)\phi^i_s \quad \text{or} \quad v^i_s = v^i_N - (z^i_o - h^i_z)\phi^i_s \; ; \tag{2-22}$$

$$u^i_N = u^i_s - (h^i_z - z^i_o)\frac{dw^i_s}{dx} - (h^i_y - y^i_o)\frac{dv^i_s}{dx} - \omega^i_{(s_N)}\frac{d\phi^i_s}{dx} \tag{2-23}$$

$$\text{or} \quad u^i_s = u^i_N + (h^i_z - z^i_o)\frac{dw^i_s}{dx} + (h^i_y - y^i_o)\frac{dv^i_s}{dx} + \omega^i_{(s_N)}\frac{d\phi^i_s}{dx} \; .$$

The difference of the stress couples along the edges of two adjacent panels is the external twist moment on the stringer. The difference of the in-plane stress resultants, shearing stress resultants and the effective transverse shear stress resultants are the external loads on stringers, i.e.,

$$V^i_{ys}(x) = V^i_y(x, y_i) - V^{i-1}_y(x, y_i) \tag{2-24}$$

$$N^i_{xys}(x) = N^i_{xy}(x, y_i) - N^{i-1}_{xy}(x, y_i) \tag{2-25}$$

$$N^i_{ys}(x) = N^i_y(x, y_i) - N^{i-1}_y(x, y_i) \tag{2-26}$$

$$M^i_{ys(x)} = M^i_{y(x,y_i)} - M^{i-1}_{y(x,y_i)} \tag{2-27}$$

By substituting Eqs.(2-21) through (2-27) into Eqs.(2-13) through (2-16), in conjunction with Eqs.(2-17) through (2-20), one obtains the following conditions which must be satisfied along $y = y_i$;

$$w^i_{(x,y_i)} = w^{i-1}_{(x,y_i)} \tag{2-28}$$

$$u^i_{(x,y_i)} = u^{i-1}_{(x,y_i)} \tag{2-29}$$

$$v^i_{(x,y_i)} = v^{i-1}_{(x,y_i)} \tag{2-30}$$

$$\left.\frac{\partial w^i}{\partial y}\right|_{y=y_i} = \left.\frac{\partial w^{i-1}}{\partial y}\right|_{y=y_i} \tag{2-31}$$

$$\left.E^i_s I^i_y \frac{\partial^4 w^i}{\partial x^4}\right|_{y=y_i} + \left.N^i_o \frac{\partial^2 w^i}{\partial x^2}\right|_{y=y_i} + \left.E^i_s I^i_{yz} \frac{\partial^4 v^i}{\partial x^4}\right|_{y=y_i} + \left.\left[E^i_s I^i_y (y^i_o - h^i_y) - E^i_s I^i_{yz} (z^i_o - h^i_z)\right] \frac{\partial^5 w^i}{\partial x^4 \partial y}\right|_{y=y_i} \tag{2-32}$$

$$\left.- N^i_o h^i_y \frac{\partial^3 w^i}{\partial x^2 \partial y}\right|_{y=y_i} = V^i_{y(x,y_i)} - V^{i-1}_{y(x,y_i)} + \left.h^i_z \frac{\partial N^i_{xy}}{\partial x}\right|_{y=y_i} - \left.h^i_z \frac{\partial N^{i-1}_{xy}}{\partial x}\right|_{y=y_i}$$

$$\left.E^i_s I^i_z \frac{\partial^4 v^i}{\partial x^4}\right|_{y=y_i} + N^i_o \frac{\partial^2 v^i}{\partial x^2} + E^i_s I^i_{yz} \frac{\partial^4 w^i}{\partial x^4} + \left.\left[E^i_s I^i_{yz} (y^i_o - h^i_y) - E^i_s I^i_z (z^i_o - h^i_z)\right] \frac{\partial^5 w^i}{\partial x^4 \partial y}\right|_{y=y_i} \tag{2-33}$$

$$\left.+ N^i_o h^i_z \frac{\partial^3 w^i}{\partial x^2 \partial y}\right|_{y=y_i} = N^i_{y(x,y_i)} - N^{i-1}_{y(x,y_i)} + \left.h^i_y \frac{\partial N^i_{xy}}{\partial x}\right|_{y=y_i} - \left.h^i_y \frac{\partial N^{i-1}_{xy}}{\partial x}\right|_{y=y_i}$$

$$- C^i_1 \frac{\partial^5 w^i}{\partial x^4 \partial y} + \left.\left\{C^i + \left[-\frac{I^i_o}{A^i_s} + (y^i_o - h^i_y) y^i_o + (z^i_o - h^i_z) z^i_o\right] N^i_o\right\} \frac{\partial^3 w^i}{\partial x^2 \partial y}\right|_{y=y_i} \tag{2-34}$$

$$\left.+ N^i_o y^i_o \frac{\partial^2 w^i}{\partial x^2}\right|_{y=y_i} - \left.N^i_o z^i_o \frac{\partial^2 v^i}{\partial x^2}\right|_{y=y_i} = M^i_{y(x,y_i)} - M^{i-1}_{y(x,y_i)} + (y^i_o - h^i_y)\left[V^i_{y(x,y_i)} - V^{i-1}_{y(x,y_i)}\right]$$

$$- (z_0^i - h_z^i)\left[N_{y(x,y_i)}^{i} - N_{y(x,y_i)}^{i-1}\right] + \omega_{(s_N)}^{i} \left.\frac{\partial N_{xy}^{i}}{\partial x}\right|_{y=y_i} - \omega_{(s_N)}^{i} \left.\frac{\partial N_{xy}^{i-1}}{\partial x}\right|_{y=y_i} ,$$

$$-E_s^i A_s^i \left\{ \frac{\partial^2 u^i}{\partial x^2} + h_z^i \frac{\partial^3 w^i}{\partial x^3} + h_y^i \frac{\partial^3 v^i}{\partial x^3} + \left[h_z^i (y_0^i - h_y^i) - h_y^i (z_0^i - h_z^i) + 2\omega_{(s_N)}^{i} \right] \left.\frac{\partial^4 w^i}{\partial x^3 \partial y}\right|_{y=y_i} \right\} \qquad (2\text{-}35)$$

$$= N_{xy(x,y_i)}^{i} - N_{xy(x,y_i)}^{i-1}$$

for i = 1, 2 . . . N, where N is the total number of stringers.

Eq.(2-35) represents the equilibrium of the stringer in the longitudinal direction. Many previous investigators have related the longitudinal force in the stringer to the average longitudinal displacement. As a result, the second and third terms in the left hand side of the equation were omitted. However, numerical computations indicate that these two terms have significant effects for some cases. Therefore, the exact form of displacement is considered throughout the present analysis. Numerical results according to the approximation have been made and will be presented later in Chapter V for comparison purposes only.

Eqs.(2-28) through (2-31) are referred to as "continuity conditions", and Eqs.(2-32) through (2-35) are referred to as "compatibility conditions". The axial force, N_0^i, obtained by considering that the stringer and the skin are having the same longitudinal strain will be used in the analysis, i.e.,

$$N_o^i = A_s^i p \frac{E_s^i}{E} ,$$

where p is the average longitudinal compressive stress in the skin.

Non-dimensionalization

For convenience, one can non-dimensionalize the previous equations by introducing the following non-dimensional quantities:

$$\xi = \frac{x}{a} \quad , \quad \eta = \frac{y}{a} \quad , \quad \zeta = \frac{z}{a} \quad , \quad \bar{y}_o^i = \frac{y_o^i}{a} \quad , \qquad (2\text{-}36)$$

$$\bar{z}_o^i = \frac{z_o^i}{a} \quad , \quad \bar{h}_y^i = \frac{h_y^i}{a} \quad , \quad \bar{h}_z^i = \frac{h_z^i}{a} \quad , \quad L = \frac{\ell}{a} \quad ,$$

$$\bar{w}^i = \frac{w^i}{a} \quad , \quad \bar{u}^i = \frac{u^i}{a} \quad , \quad \bar{v}^i = \frac{v^i}{a} \quad , \quad \bar{q} = \frac{q}{E}\left(\frac{a}{h}\right) \quad ,$$

$$\bar{q}_x = \frac{q_x}{E}\left(\frac{a}{h}\right) \quad , \quad \bar{q}_y = \frac{q_y}{E}\left(\frac{a}{h}\right) \quad , \quad \rho^2 = 12(1-\nu^2)\left(\frac{a}{h}\right)^2 \quad ,$$

$$k_1^i = \frac{I_{yz}^i}{I_y^i} \quad , \quad k_2^i = \frac{I_{yz}^i}{I_z^i} \quad , \quad k_3 = 12\left(\frac{a}{h}\right)^2 \quad ,$$

$$\bar{A}_1^i = 12(1-\nu^2)\,\frac{E_s^i}{E}\,\frac{A_s^i}{ah}\left(\frac{a}{h}\right)^2 \quad , \quad \bar{A}_2^i = (1-\nu^2)\,\frac{E_s^i}{E}\,\frac{A_s^i}{ah} \quad ,$$

$$\bar{I}_y^i = 12(1-\nu^2)\,\frac{E_s^i}{E}\,\frac{A_s^i}{ah^3} \quad , \quad \bar{I}_z^i = (1-\nu^2)\,\frac{E_s^i}{E}\,\frac{I_z^i}{a^3h} \quad ,$$

$$\bar{C}_1^i = 12(1-\nu^2)\,\frac{C_1^i}{Ea^3h^3} \quad , \quad \bar{C}^i = 12(1-\nu^2)\,\frac{C^i}{Eah^3} \quad ,$$

$$\bar{I}_o^i = \frac{I_o^i}{A_s^i\, a^2} \quad , \quad \bar{N}_o^i = A_1^i\left(\frac{p}{E}\right) \quad , \quad \bar{\bar{N}}_o^i = A_2^i\left(\frac{p}{E}\right) \; ,$$

$$\bar{M}_x^i = \frac{12(1-\nu^2)a}{Eh^3}\, M_x^i \quad , \quad \bar{M}_y^i = \frac{12(1-\nu^2)a}{Eh^3}\, M_y^i \quad ,$$

$$\bar{M}_{xy}^i = \frac{12(1-\nu^2)a}{Eh^3}\, M_{xy}^i \; , \quad \bar{V}_x^i = \frac{12(1-\nu^2)a^2}{Eh^3}\, V_x^i \; ,$$

$$\bar{V}_y^i = \frac{12(1-\nu^2)a^2}{Eh^3}\, V_y^i \quad , \quad \bar{N}_x^i = \frac{(1-\nu^2)}{Eh}\, N_x^i \quad ,$$

$$\bar{N}_y^i = \frac{(1-\nu^2)}{Eh}\, N_y^i \quad , \quad \bar{N}_{xy}^i = \frac{(1-\nu^2)}{Eh}\, N_{xy}^i \; ,$$

$$\bar{\omega}_N^i = \frac{\omega^i(s_N)}{a^2} \quad , \quad \sigma = \left(\frac{1}{2}\,\frac{p}{E}\right)^{\frac{1}{2}} \quad .$$

Then Eqs.(2-2) through (2-12) and Eqs.(2-28) through (2-35) can be expressed in the nondimensional form as follows:

$$\frac{1}{\rho^2}\bar{\nabla}^8\bar{w} + \frac{\partial^4\bar{w}}{\partial\xi^4} - \frac{1}{1-\nu^2}\bar{\nabla}^4\Big[\frac{\partial}{\partial\xi}\Big(\bar{N}_x\frac{\partial\bar{w}}{\partial\xi}\Big) + \frac{\partial}{\partial\xi}\Big(\bar{N}_{xy}\frac{\partial\bar{w}}{\partial\eta}\Big) \tag{2-37}$$

$$+ \frac{\partial}{\partial\eta}\Big(\bar{N}_{xy}\frac{\partial\bar{w}}{\partial\xi}\Big) + \frac{\partial}{\partial\eta}\Big(\bar{N}_y\frac{\partial\bar{w}}{\partial\eta}\Big)\Big] = \bar{\nabla}^4\bar{q} - \Big[\nu\frac{\partial^3\bar{q}_x}{\partial\xi^3} - \frac{\partial^3\bar{q}_x}{\partial\xi\partial\eta^2}$$

$$+ \frac{\partial^3\bar{q}_y}{\partial\eta^3} + (2+\nu)\frac{\partial^3\bar{q}_y}{\partial\xi^2\partial\eta}\Big] + \frac{\partial^2}{\partial\xi^2}\Big[\Big(\frac{\partial^2\bar{w}}{\partial\xi\partial\eta}\Big)^2 - \frac{\partial^2\bar{w}}{\partial\xi^2}\frac{\partial^2\bar{w}}{\partial\eta^2}\Big]$$

$$\bar{\nabla}^4 \bar{u} = \nu \frac{\partial^3 \bar{w}}{\partial \xi^3} - \frac{\partial^3 \bar{w}}{\partial \xi \partial \eta^2} - (1-\nu^2)\left(\frac{\partial^2 \bar{q}_x}{\partial \xi^2} + \frac{2}{1-\nu}\frac{\partial^2 \bar{q}_x}{\partial \eta^2} - \frac{1+\nu}{1-\nu}\frac{\partial^2 \bar{q}_y}{\partial \xi \partial \eta} \right) , \tag{2-38}$$

$$\bar{\nabla}^4 \bar{v} = (2+\nu)\frac{\partial^3 \bar{w}}{\partial \xi^2 \partial \eta} + \frac{\partial^3 \bar{w}}{\partial \eta^3} - (1-\nu^2)\left(\frac{\partial^2 \bar{q}_y}{\partial \eta^2} + \frac{2}{1-\nu}\frac{\partial^2 \bar{q}_y}{\partial \xi^2} - \frac{1+\nu}{1-\nu}\frac{\partial^2 \bar{q}_x}{\partial \xi \partial \eta} \right) , \tag{2-39}$$

$$\bar{N}_x = \frac{\partial \bar{u}}{\partial \xi} + \nu\left(\frac{\partial \bar{v}}{\partial \eta} - \bar{w} \right) , \tag{2-40}$$

$$\bar{N}_y = \frac{\partial \bar{v}}{\partial \eta} - \bar{w} + \nu \frac{\partial \bar{u}}{\partial \xi} , \tag{2-41}$$

$$\bar{N}_{xy} = \frac{1}{2}(1-\nu)\left(\frac{\partial \bar{u}}{\partial \eta} + \frac{\partial \bar{v}}{\partial \xi} \right) , \tag{2-42}$$

$$V_x = -\left(\frac{\partial^3 \bar{w}}{\partial \xi^3} + (2-\nu)\frac{\partial^3 \bar{w}}{\partial \xi \partial \eta^2} \right) , \tag{2-43}$$

$$V_y = -\left(\frac{\partial^3 \bar{w}}{\partial \eta^3} + (2-\nu)\frac{\partial^3 \bar{w}}{\partial \xi^2 \partial \eta} \right) , \tag{2-44}$$

$$\bar{M}_x = -\left(\frac{\partial^2 \bar{w}}{\partial \xi^2} + \nu \frac{\partial^2 \bar{w}}{\partial \eta^2} \right) , \tag{2-45}$$

$$\bar{M}_y = -\left(\frac{\partial^2 \bar{w}}{\partial \eta^2} + \nu \frac{\partial^2 \bar{w}}{\partial \xi^2} \right) , \tag{2-46}$$

$$\bar{M}_{xy} = -(1-\nu)\frac{\partial^2 \bar{w}}{\partial \xi \partial \eta} , \tag{2-47}$$

$$\bar{w}^i_{(\xi,\eta_i)} = \bar{w}^{i-1}_{(\xi,\eta_i)} , \tag{2-48}$$

$$\bar{u}^i_{(\xi,\eta_i)} = \bar{u}^{i-1}_{(\xi,\eta_i)} , \tag{2-49}$$

$$\bar{v}^i_{(\xi,\eta_i)} = \bar{v}^{i-1}_{(\xi,\eta_i)} , \tag{2-50}$$

$$\left.\frac{\partial \bar{w}^i}{\partial \eta}\right|_{\eta=\eta_i} = \left.\frac{\partial \bar{w}^{i-1}}{\partial \eta}\right|_{\eta=\eta_i} , \tag{2-51}$$

$$\bar{I}_y^i \frac{\partial^4 \bar{w}^i}{\partial \xi^4}\bigg|_{\eta=\eta_i} + \bar{N}_o^i \frac{\partial^2 \bar{w}^i}{\partial \xi^2}\bigg|_{\eta=\eta_i} + K_1^i \bar{I}_y^i \frac{\partial^4 \bar{v}^i}{\partial \xi^4}\bigg|_{\eta=\eta_i} + \bar{d}_1^i \bar{I}_y^i \frac{\partial^5 \bar{w}^i}{\partial \xi^4 \partial \eta}\bigg|_{\eta=\eta_i} - \bar{N}_o^i \bar{h}_y^i \frac{\partial^3 \bar{w}^i}{\partial \xi^2 \partial \eta}\bigg|_{\eta=\eta_i} \tag{2-52}$$

$$= \bar{V}_y^i(\xi,\eta_i) - \bar{V}_y^{i-1}(\xi,\eta_i) + K_3 \bar{h}_z^i \frac{\partial \bar{N}_{xy}^i}{\partial \xi}\bigg|_{\eta=\eta_i} - K_3 \bar{h}_z^i \frac{\partial \bar{N}_{xy}^{i-1}}{\partial \xi}\bigg|_{\eta=\eta_i} ,$$

$$\bar{I}_z^i \frac{\partial^4 \bar{v}^i}{\partial \xi^4}\bigg|_{\eta=\eta_i} + \bar{N}_o^i \frac{\partial^2 \bar{v}^i}{\partial \xi^2}\bigg|_{\eta=\eta_i} + K_2 \bar{I}_z^i \frac{\partial^4 \bar{w}^i}{\partial \xi^4}\bigg|_{\eta=\eta_i} + \bar{d}_2^i \bar{I}_z^i \frac{\partial^5 \bar{w}^i}{\partial \xi^4 \partial \eta}\bigg|_{\eta=\eta_i} + \bar{N}_o^i \bar{h}_z^i \frac{\partial^3 \bar{w}^i}{\partial \xi^2 \partial \eta}\bigg|_{\eta=\eta_i} \tag{2-53}$$

$$= \bar{N}_y^i(\xi,\eta_i) - \bar{N}_y^{i-1}(\xi,\eta_i) + \bar{h}_y^i \frac{\partial \bar{N}_{xy}^i}{\partial \xi}\bigg|_{\eta=\eta_i} - \bar{h}_y^i \frac{\partial \bar{N}^{i-1}}{\partial \xi}\bigg|_{\eta=\eta_i} ,$$

$$-\bar{c}_1^i \frac{\partial^5 \bar{w}^i}{\partial \xi^4 \partial \eta}\bigg|_{\eta=\eta_i} + \bar{c}_o^i \frac{\partial^3 \bar{w}^i}{\partial \xi^2 \partial \eta}\bigg|_{\eta=\eta_i} + \bar{N}_o^i \bar{y}_o^i \frac{\partial^2 \bar{w}^i}{\partial \xi^2}\bigg|_{\eta=\eta_i} - \bar{N}_o^i \bar{z}_o \frac{\partial^2 \bar{v}^i}{\partial \xi^2}\bigg|_{\eta=\eta_i} \tag{2-54}$$

$$= \bar{M}_y^i(\xi,\eta_i) - \bar{M}_y^{i-1}(\xi,\eta_i) + (\bar{y}_o^i - \bar{h}_y^i)\bar{V}_y^i(\xi,\eta_i) - (\bar{y}_o^i - \bar{h}_y^i)\bar{V}_y^{i-1}(\xi,\eta_i)$$

$$- K_3(\bar{z}_o^i - \bar{h}_z^i)\bar{N}_y^i(\xi,\eta_i) + K_3(\bar{z}_o^i - \bar{h}_z^i)\bar{N}^{i-1}(\xi,\eta_i) + K_3 \bar{\omega}_N^i \frac{\partial \bar{N}_{xy}^i}{\partial \xi}\bigg|_{\eta=\eta_i} - K \bar{\omega}_N^i \frac{\partial \bar{N}^{i-1}}{\partial \xi}\bigg|_{\eta=\eta_i} ,$$

$$-\bar{A}_2^i \left(\frac{\partial^2 \bar{u}^i}{\partial \xi^2} + \bar{h}_z^i \frac{\partial^3 \bar{w}^i}{\partial \xi^3} + \bar{h}_y^i \frac{\partial^3 \bar{v}^i}{\partial \xi^3} + \bar{d}_3^i \frac{\partial^4 \bar{w}^i}{\partial \xi^3 \partial \eta} \right)\bigg|_{\eta=\eta_i} = \bar{N}_{xy}^i(\xi,\eta_i) - \bar{N}_{xy}^{i-1}(\xi,\eta_i) \tag{2-55}$$

where

$$\bar{\nabla}^2 = \frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2} , \quad \bar{d}_1^i = (\bar{y}_o^i - \bar{h}_y^i) - K_1^i(\bar{z}_o^i - \bar{h}_z^i),$$

$$\bar{d}_2^i = K_2^i(\bar{y}_o^i - \bar{h}_y^i) - (\bar{z}_o^i - \bar{h}_z^i) , \quad \bar{d}_3^i = \bar{h}_z^i(\bar{y}_o^i - \bar{h}_y^i) - \bar{h}_y^i(\bar{z}_o^i - \bar{h}_z^i) + \bar{\omega}_N^i ,$$

$$\bar{c}_o^i = \bar{c}^i - \left[\bar{I}_o^i - \bar{y}_o^i(\bar{y}_o^i - \bar{h}_y^i) - \bar{z}_o^i(\bar{z}_o^i - \bar{h}_z^i) \right] \bar{N}_o^i .$$

CHAPTER III

LINEAR STABILITY ANALYSIS OF SIMPLY SUPPORTED STIFFENED SHELLS

If, in the governing differential equation (Eq.(2-37)), the non-linear terms are considered to be small in comparison with the other terms and hence neglected, the linear equation of Donnell-type is yielded. Furthermore, for the case of axial compression, $q_x = q_y = q = 0$. Assume that the boundary conditions are such that in-plane stretching forces can be obtained according to membrane analysis, so that $N_x = -ph$ and $N_y = N_{xy} = 0$, where p is the applied axial force per unit area of skin. Then Eq.(2-37) can be linearized and written as

$$\frac{1}{\rho^2}\bar{\nabla}^8\bar{w} + 2\sigma^2\frac{\partial^2}{\partial\xi^2}(\bar{\nabla}^4\bar{w}) + \frac{\partial^4\bar{w}}{\partial\xi^4} = 0 \ , \tag{3-1}$$

where

$$\sigma = \left(\frac{1}{2}\frac{p}{E}\right)^{1/2}.$$

In the linear case, the buckling increments are uncoupled from the pre-buckling deformations, and Eqs.(2-37) through (2-55) and Eq.(3-1) are written in terms of the perturbation quantities measured from the pre-buckling deformations.

For the simply supported case, the classical boundary conditions yield $w = v = N_x = M_x = 0$ at both ends ($x = 0, \ell$). If the form of the solution of Eq.(3-1) is assumed to be

$$\bar{w}(\xi,\eta) = \sum_{m=1}^{\infty} W_m(\eta) \sin \frac{m\pi\xi}{L} , \tag{3-2'}$$

then the classical boundary conditions for the simply supported case can be satisfied.

However, solutions of Eqs.(3-1), (2-38) and (2-39), corresponding to the form of Eq.(3-2'), are in the form of trignometric series. Since $\sin\frac{m\pi\xi}{L}$ and $\cos\frac{m\pi\xi}{L}$ are orthogonal functions, the coefficients of each trigonometric function must be zero if the whole series is zero for any value of ξ. It can be shown that only the single sinusoidal function is necessary in Eq.(3-2'). As a result, the form of the solution of Eq.(3-1) becomes

$$\bar{w}(\xi,\eta) = W_m(\eta) \sin\frac{m\pi\xi}{L} , \tag{3-2}$$

<u>Solution for Displacement Components u and v</u>

In order to avoid solving Eqs.(2-38) and (2-39) for displacement components u and v explicitly, Eq.(3-1) can be written in the form

$$\bar{\nabla}^4\left(\frac{1}{\rho^2}\bar{\nabla}^4 + 2\sigma^2\left(\frac{\partial}{\partial\xi}\right)^2 \right)\bar{w} = -\left(\frac{\partial}{\partial\xi}\right)^4\bar{w} \tag{3-3}$$

If negative exponents are introduced for the differential operators and are defined to denote inverse operations such that

$$\left(\frac{\partial}{\partial\xi}\right)^{-1}\left(\frac{\partial}{\partial\xi}\right)\bar{w} = \bar{w}$$

and

$$\bar{\nabla}^{-4}\left(\bar{\nabla}^4\bar{w}\right) = \bar{w}$$

the radial displacement can, from Eq.(3-3), be written symbolically as

$$\bar{w} = -\bar{\nabla}^4\left[\frac{1}{\rho^2}\left(\frac{\partial}{\partial\xi}\right)^{-4}\bar{\nabla}^4 + 2\sigma^2\left(\frac{\partial}{\partial\xi}\right)^{-2}\right]\bar{w}. \tag{3-4}$$

Substitution of Eq.(3-4) into the right hand member of Eq.(2-38) results in

$$\bar{\nabla}^4\bar{u} = -\bar{\nabla}^4\left\{\frac{1}{\rho^2}\left[\nu\left(\frac{\partial}{\partial\xi}\right)^{-1}\bar{\nabla}^4 - \left(\frac{\partial}{\partial\xi}\right)^{-3}\left(\frac{\partial}{\partial\eta}\right)^{2}\bar{\nabla}^4\right] + 2\sigma^2\left[\nu\left(\frac{\partial}{\partial\xi}\right) - \left(\frac{\partial}{\partial\xi}\right)^{-1}\left(\frac{\partial}{\partial\eta}\right)^{2}\right]\right\}\bar{w}. \tag{3-5}$$

Hence

$$\bar{u} = \left\{\frac{1}{\rho^2}\left[\left(\frac{\partial}{\partial\eta}\right)^{2}\left(\frac{\partial}{\partial\xi}\right)^{-3}\bar{\nabla}^4 - \nu\left(\frac{\partial}{\partial\xi}\right)^{-1}\bar{\nabla}^4\right] + 2\sigma^2\left[\left(\frac{\partial}{\partial\eta}\right)^{2}\left(\frac{\partial}{\partial\xi}\right)^{-1} - \nu\left(\frac{\partial}{\partial\xi}\right)\right]\right\}\bar{w}. \tag{3-6}$$

Similarly from Eq.(2-39) one obtains

$$\bar{v} = -\left\{\frac{1}{\rho^2}\left[\left(\frac{\partial}{\partial\eta}\right)^{3}\left(\frac{\partial}{\partial\xi}\right)^{-4}\bar{\nabla}^4 + (2+\nu)\left(\frac{\partial}{\partial\xi}\right)^{-2}\left(\frac{\partial}{\partial\eta}\right)\bar{\nabla}^4\right] + 2\sigma^2\left[\left(\frac{\partial}{\partial\xi}\right)^{-2}\left(\frac{\partial}{\partial\eta}\right)^{3} + (2+\nu)\left(\frac{\partial}{\partial\eta}\right)\right]\right\}\bar{w}. \tag{3-7}$$

Substitution of Eq.(3-2) into Eqs.(3-6) and (3-7), respectively, results in expressions for the displacement components u and v as follows:

$$\bar{u} = U_m(\eta)\cos\frac{m\pi\xi}{L}, \tag{3-8}$$

where

$$U_m(\eta) = \left\{\frac{1}{\rho^2}\left[\left(\frac{m\pi}{L}\right)^{-3}\frac{d^6}{d\eta^6} + \left(\frac{m\pi}{L}\right)^{-1}(\nu-2)\frac{d^4}{d\eta^4} + \left(\frac{m\pi}{L}\right)(1-2\nu)\frac{d^2}{d\eta^2} + \left(\frac{m\pi}{L}\right)^{3}\nu\right]\right.$$

$$\left. - 2\sigma^2\left[\left(\frac{m\pi}{L}\right)^{-1}\frac{d^2}{d\eta^2} + \nu\left(\frac{m\pi}{L}\right)\right]\right\}W_m(\eta),$$

and

$$\bar{v} = V_m(\eta)\ \sin\frac{m\pi\xi}{L} \tag{3-9}$$

where

$$V_m(\eta) = \frac{d}{d\eta}\left\{-\frac{1}{\rho^2}\left[\left(\frac{m\pi}{L}\right)^{-4}\frac{d^6}{d\eta^6} - \left(\frac{m\pi}{L}\right)^{-2}\frac{d^4}{d\eta^4} + (5+2\nu)\frac{d^2}{d\eta^2} - \left(\frac{m\pi}{L}\right)^2(2+\nu)\right]\right.$$

$$\left. + 2\sigma^2\left[\left(\frac{m\pi}{L}\right)^{-2}\frac{d^2}{d\eta^2} - (2+\nu)\right]\right\}W_m(\eta).$$

Solution for Displacement Component w

By substituting Eq.(3-2) into Eq.(3-1), the following ordinary differential equation is obtained,

$$\frac{1}{\rho^2}\left[\frac{d^2}{d\eta^2} - \left(\frac{m\pi}{L}\right)^2\right]^4 W_m(\eta) - 2\sigma^2\left(\frac{m\pi}{L}\right)^2\left[\frac{d^2}{d\eta^2} - \left(\frac{m\pi}{L}\right)^2\right]^2 W_m(\eta) + \left(\frac{m\pi}{L}\right)^4 W_m(\eta) = 0. \tag{3-10}$$

Let the solution be $Ae^{\lambda_m\eta}$ to obtain the characteristic equation

$$\left[\lambda_m^2 - \left(\frac{m\pi}{L}\right)^2\right]^4 - 2\sigma^2\rho^2\left(\frac{m\pi}{L}\right)^2\left[\lambda_m^2 - \left(\frac{m\pi}{L}\right)^2\right]^2 + \rho^2\left(\frac{m\pi}{L}\right)^4 = 0. \tag{3-11}$$

Let the term $\left[\lambda_m^2 - \left(\frac{m\pi}{L}\right)^2\right]^2$ be an independent variable. Eq.(3-11) then becomes a quardratic equation which can be solved explicitly as

$$\left[\lambda_m^2 - \left(\frac{m\pi}{L}\right)^2\right]^2 = \rho^2\sigma^2\left(\frac{m\pi}{L}\right)^2\left[1 \pm \left(1 - \frac{1}{\rho^2\sigma^4}\right)^{1/2}\right]. \tag{3-12}$$

If $\rho^2\sigma^4 < 1$, then all roots are complex: otherwise, they are real and/or pure imaginary. There are four different forms of solutions:

(I) $\rho\sigma^2 \geq 1$:

If $\rho\sigma^2 \geq 1$, then $\left(1 - \frac{1}{\rho^2\sigma^4}\right) \geq 0$, and

$$\lambda_m^2 = \left(\frac{m\pi}{L}\right)\left\{\left(\frac{m\pi}{L}\right) \pm \rho\sigma\left[1 \pm \left(1 - \frac{1}{\rho^2\sigma^4}\right)^{1/2}\right]^{1/2}\right\}. \tag{3-13}$$

Since $0 < \frac{1}{\rho^2\sigma^4} \leq 1$, which implies $\left(1 - \frac{1}{\rho^2\sigma^4}\right)^{1/2} \leq 1$, hence

$$1 \pm \left(1 - \frac{1}{\rho^2\sigma^4}\right)^{1/2} \geq 0.$$

(i) If $\rho\sigma\left[1 - \left(1 - \frac{1}{\rho^2\sigma^4}\right)^{1/2}\right]^{1/2} \geq \left(\frac{m\pi}{L}\right)$, then

$$\left(\frac{m\pi}{L}\right) - \rho\sigma\left[1 \pm \left(1 - \frac{1}{\rho^2\sigma^4}\right)^{1/2}\right]^{1/2} \leq 0,$$

and there are four real roots and four pure imaginary roots (in this case $R_{1m} = R_{2m} = 2$, $R_{3m} = 0$, where $2R_{1m}$, $2R_{2m}$ and $4R_{3m}$ are the numbers of real, pure imaginary and complex roots, respectively):

$$\lambda_{m_{1,2}} = \pm \alpha_{m1} \quad ; \quad \lambda_{m_{3,4}} = \pm \alpha_{m2} \quad ;$$

$$\lambda_{m_{5,6}} = \pm i\beta_{m1} \quad ; \quad \lambda_{m_{7,8}} = \pm i\beta_{m2} \quad ,$$

where

$$\alpha_{m1} = \left(\frac{m\pi}{L}\right)^{1/2}\left\{\frac{m\pi}{L} + \rho\sigma\left[1 + \left(1 - \frac{1}{\rho^2\sigma^4}\right)^{1/2}\right]^{1/2}\right\}^{1/2}$$

$$\alpha_{m2} = \left(\frac{m\pi}{L}\right)^{1/2}\left\{\frac{m\pi}{L} + \rho\sigma\left[1 - \left(1 - \frac{1}{\rho^2\sigma^4}\right)^{1/2}\right]^{1/2}\right\}^{1/2}$$

$$\beta_{m1} = \left(\frac{m\pi}{L}\right)^{1/2}\left\{\rho\sigma\left[1 + \left(1 - \frac{1}{\rho^2\sigma^4}\right)^{1/2}\right]^{1/2} - \frac{m\pi}{L}\right\}^{1/2}$$

$$\beta_{m2} = \left(\frac{m\pi}{L}\right)^{1/2}\left\{\rho\sigma\left[1 - \left(1 - \frac{1}{\rho^2\sigma^4}\right)^{1/2}\right]^{1/2} - \frac{m\pi}{L}\right\}^{1/2}$$

Hence, the solution for Eq.(3-12) is

$$W_m(\eta) = A_1 e^{-\alpha_{m1}\eta} + A_2 e^{-\alpha_{m2}\eta} + A_3 e^{\alpha_{m1}\eta} + A_4 e^{\alpha_{m2}\eta} \tag{3-14}$$

$$+ A_5 \cos\beta_{m1}\eta + A_6 \cos\beta_{m2}\eta + A_7 \sin\beta_{m1}\eta + A_8 \sin\beta_{m2}\eta \ .$$

(ii) If $\rho\sigma\left[1 + \left(1 - \frac{1}{\rho^2\sigma^4}\right)^{1/2}\right]^{1/2} \leq \frac{m\pi}{L}$, then

$$\left(\frac{m\pi}{L}\right) \pm \rho\sigma\left[1 \pm \left(1 - \frac{1}{\rho^2\sigma^4}\right)^{1/2}\right]^{1/2} \geq 0 ,$$

and all roots are real (in this case $R_{1m} = 4$, $R_{2m} = R_{3m} = 0$):

$$\lambda_{m\,1,2} = \pm\alpha_{m1} \ ; \qquad \lambda_{m\,3,4} = \pm\alpha_{m2} \ ;$$

$$\lambda_{m\,5,6} = \pm\alpha_{m3} \ ; \qquad \lambda_{m\,7,8} = \pm\alpha_{m4} \ ,$$

where

$$\alpha_{m1} = \left(\frac{m\pi}{L}\right)^{1/2}\left\{\frac{m\pi}{L} + \rho\sigma\left[1 + \left(1 - \frac{1}{\rho^2\sigma^4}\right)^{1/2}\right]^{1/2}\right\}^{1/2} \ ;$$

$$\alpha_{m2} = \left(\frac{m\pi}{L}\right)^{1/2}\left\{\frac{m\pi}{L} + \rho\sigma\left[1 - \left(1 - \frac{1}{\rho^2\sigma^4}\right)^{1/2}\right]^{1/2}\right\}^{1/2} \ ;$$

$$\alpha_{m3} = \left(\frac{m\pi}{L}\right)^{1/2}\left\{\frac{m\pi}{L} - \rho\sigma\left[1 - \left(1 - \frac{1}{\rho^2\sigma^4}\right)^{1/2}\right]^{1/2}\right\}^{1/2} \ ;$$

$$\alpha_{m4} = \left(\frac{m\pi}{L}\right)^{1/2}\left\{\frac{m\pi}{L} - \rho\sigma\left[1 + \left(1 - \frac{1}{\rho^2\sigma^4}\right)^{1/2}\right]^{1/2}\right\}^{1/2} .$$

Hence, the solution for Eq.(3-12) is

$$W_m(\eta) = A_1 e^{-\alpha_{m1}\eta} + A_2 e^{-\alpha_{m2}\eta} + A_3 e^{-\alpha_{m3}\eta} + A_4 e^{-\alpha_{m4}\eta} \tag{3-15}$$

$$+ A_5 e^{\alpha_{m1}\eta} + A_6 e^{\alpha_{m2}\eta} + A_7 e^{\alpha_{m3}\eta} + A_8 e^{\alpha_{m4}\eta} .$$

(iii) If $\rho\sigma\left[1 - (1 - \frac{1}{\rho^2\sigma^4})^{1/2}\right]^{1/2} \leq \frac{m\pi}{L}$, but $\rho\sigma\left[1 + (1 - \frac{1}{\rho^2\sigma^4})^{1/2}\right]^{1/2} \geq \frac{m\pi}{L}$, then only one pair of roots are pure imaginary (conjugate), and the rest of them are real (in this case $R_{1m} = 3$, $R_{2m} = 1$, $R_{3m} = 0$):

$$\lambda_{m_{1,2}} = \pm \alpha_{m1} \quad ; \quad \lambda_{m_{3,4}} = \pm \alpha_{m2} \quad ;$$

$$\lambda_{m_{5,6}} = \pm \alpha_{m3} \quad ; \quad \lambda_{m_{7,8}} = \pm i\beta_{m1} \quad ,$$

where

$$\alpha_{m1} = \left(\frac{m\pi}{L}\right)^{1/2}\left\{\frac{m\pi}{L} + \rho\sigma\left[1 + (1 - \frac{1}{\rho^2\sigma^4})^{1/2}\right]^{1/2}\right\}^{1/2} ;$$

$$\alpha_{m2} = \left(\frac{m\pi}{L}\right)^{1/2}\left\{\frac{m\pi}{L} + \rho\sigma\left[1 - (1 - \frac{1}{\rho^2\sigma^4})^{1/2}\right]^{1/2}\right\}^{1/2} ;$$

$$\alpha_{m3} = \left(\frac{m\pi}{L}\right)^{1/2}\left\{\frac{m\pi}{L} - \rho\sigma\left[1 - (1 - \frac{1}{\rho^2\sigma^4})^{1/2}\right]^{1/2}\right\}^{1/2} ;$$

$$\beta_{m1} = \left(\frac{m\pi}{L}\right)^{1/2}\left\{\rho\sigma\left[1 + (1 - \frac{1}{\rho^2\sigma^4})^{1/2}\right]^{1/2} - \frac{m\pi}{L}\right\}^{1/2} .$$

Hence, the solution for Eq.(3-12) is

$$W_m(\eta) = A_1 e^{-\alpha_{m1}\eta} + A_2 e^{-\alpha_{m2}\eta} + A_3 e^{-\alpha_{m3}\eta} + A_4 e^{\alpha_{m1}\eta} \tag{3-16}$$

$$+ A_5 e^{\alpha_{m2}\eta} + A_6 e^{\alpha_{m3}\eta} + A_7 \cos\beta_{m1}\eta + A_8 \sin\beta_{m1}\eta$$

(II) $\rho\sigma^2 < 1$:

If $\rho\sigma^2 < 1$, then $\left(1 - \frac{1}{\rho^2\sigma^4}\right) < 0$, and all roots are complex, which can be expressed as

$$\pm \gamma_{mj} \pm i\,\delta_{mj} , \qquad \text{for } j = 1, 2 ,$$

where

$$\gamma_{m1} = \frac{1}{\sqrt{2}}\left[\left(A_m^{+2} + B_m^2\right)^{1/2} + A_m^{+}\right]^{1/2} ; \quad \delta_{m1} = \frac{1}{\sqrt{2}}\left[\left(A_m^{+2} + B_m^2\right)^{1/2} - A_m^{+}\right]^{1/2} ;$$

$$\gamma_{m2} = \frac{1}{\sqrt{2}}\left[\left(A_m^{-2} + B_m^2\right)^{1/2} + A_m^{-}\right]^{1/2} ; \quad \delta_{m2} = \frac{1}{\sqrt{2}}\left[\left(A_m^{-2} + B_m^2\right)^{1/2} - A_m^{-}\right]^{1/2} ;$$

in which

$$A_m^{+} = \left(\frac{m\pi}{L}\right)^2 + \frac{1}{\sqrt{2}}\rho\sigma\frac{m\pi}{L}\left(\frac{1}{\rho\sigma^2} + 1\right)^{1/2} ,$$

$$A_m^{-} = \left(\frac{m\pi}{L}\right)^2 - \frac{1}{\sqrt{2}}\rho\sigma\frac{m\pi}{L}\left(\frac{1}{\rho\sigma^2} + 1\right)^{1/2} ,$$

$$B_m = \frac{1}{\sqrt{2}}\rho\sigma\frac{m\pi}{L}\left(\frac{1}{\rho\sigma^2} - 1\right)^{1/2} .$$

In this case, $R_{1m} = R_{2m} = 0$, $R_{3m} = 2$, and the radial displacement is

$$\begin{aligned} W_m(\eta) = {} & A_1 e^{-\gamma_{m1}\eta}\cos\delta_{m1}\eta + A_2 e^{-\gamma_{m2}\eta}\cos\delta_{m2}\eta + A_3 e^{-\gamma_{m1}\eta}\sin\delta_{m1}\eta \\ & + A_4 e^{-\gamma_{m2}\eta}\sin\delta_{m2}\eta + A_5 e^{\gamma_{m1}\eta}\cos\delta_{m1}\eta + A_6 e^{\gamma_{m2}\eta}\cos\delta_{m2}\eta \\ & + A_7 e^{\gamma_{m1}\eta}\sin\delta_{m1}\eta + A_8 e^{\gamma_{m2}\eta}\sin\delta_{m2}\eta . \end{aligned} \qquad (3\text{-}17)$$

The Stability Determinant

If it is noted that the summation is ignored, when the upper limit of the summation is zero, the expressions for Eqs.(3-14), (3-15), (3-16) and (3-17) can, for convenience, be combined; Eq.(3-2) can be written in the most general expression, for the i^{th} panel, as

$$
\begin{aligned}
\bar{w}^i_{(\xi,\eta)} = \sin\frac{m\pi\xi}{L}\Big\{ & \sum_{j=1}^{R_{1m}} \left[A^i_j e^{-\alpha_{mj}\eta} + A^i_{j+R_{1m}} e^{\alpha_{mj}\eta}\right] \\
& + \sum_{j=1}^{R_{2m}} \left[A^i_{j+a_m}\cos\beta_{mj}\eta + A^i_{j+b_m}\sin\beta_{mj}\eta\right] \\
& + \sum_{j=1}^{R_{3m}} \left[A^i_{j+c_m} e^{-\gamma_{mj}\eta}\cos\delta_{mj}\eta + A^i_{j+d_m} e^{-\gamma_{mj}\eta}\sin\delta_{mj}\eta \right. \\
& \left. + A^i_{j+e_m} e^{\gamma_{mj}\eta}\cos\delta_{mj}\eta + A^i_{j+f_m} e^{\gamma_{mj}\eta}\sin\delta_{mj}\eta\right]\Big\} ,
\end{aligned}
\tag{3-18}
$$

where

$$a_m = 2R_{1m} \quad ; \quad b_m = a_m + R_{2m} \quad ; \quad c_m = b_m + R_{2m} \; ;$$

$$d_m = c_m + R_{3m} \quad ; \quad e_m = d_m + R_{3m} \quad ; \quad f_m = e_m + R_{3m} \; ;$$

$$\alpha_{m1} = \left(\frac{m\pi}{L}\right)^{1/2}\left\{\frac{m\pi}{L} + \rho\sigma\left[1 + \left(1 - \frac{1}{\rho^2\sigma^4}\right)^{1/2}\right]^{1/2}\right\}^{1/2} ;$$

$$\alpha_{m2} = \left(\frac{m\pi}{L}\right)^{1/2}\left\{\frac{m\pi}{L} + \rho\sigma\left[1 - \left(1 - \frac{1}{\rho^2\sigma^4}\right)^{1/2}\right]^{1/2}\right\}^{1/2} ;$$

$$\alpha_{m3} = \left(\frac{m\pi}{L}\right)^{1/2}\left\{\frac{m\pi}{L} - \rho\sigma\left[1 - \left(1 - \frac{1}{\rho^2\sigma^4}\right)^{1/2}\right]^{1/2}\right\}^{1/2} ;$$

$$\alpha_{m4} = \left(\frac{m\pi}{L}\right)^{1/2}\left\{\frac{m\pi}{L} - \rho\sigma\left[1 + \left(1 - \frac{1}{\rho^2\sigma^4}\right)^{1/2}\right]^{1/2}\right\}^{1/2} ;$$

$$\beta_{m_1} = \left(\frac{m\pi}{L}\right)^{1/2}\left\{\rho\sigma\left[1 + \left(1 - \frac{1}{\rho^2\sigma^4}\right)^{1/2}\right]^{1/2} - \frac{m\pi}{L}\right\}^{1/2} \quad ;$$

$$\beta_{m_2} = \left(\frac{m\pi}{L}\right)^{1/2}\left\{\rho\sigma\left[1 - \left(1 - \frac{1}{\rho^2\sigma^4}\right)^{1/2}\right]^{1/2} - \frac{m\pi}{L}\right\}^{1/2} \quad ;$$

$$\gamma_{m_1} = \frac{1}{\sqrt{2}}\left[\left(A_m^{+2} + B_m^2\right)^{1/2} + A_m^+\right]^{1/2} \quad ; \qquad \delta_{m_1} = \frac{1}{\sqrt{2}}\left[\left(A_m^{+2} + B_m^2\right)^{1/2} - A_m^+\right]^{1/2} \; ;$$

$$\gamma_{m_2} = \frac{1}{\sqrt{2}}\left[\left(A_m^{-2} + B_m^2\right)^{1/2} + A_m^-\right]^{1/2} \quad ; \qquad \delta_{m_2} = \frac{1}{\sqrt{2}}\left[\left(A_m^{-2} + B_m^2\right)^{1/2} - A_m^-\right]^{1/2} \; ;$$

in which

$$A_m^+ = \left(\frac{m\pi}{L}\right)^2 + \frac{1}{\sqrt{2}}\rho\sigma\frac{m\pi}{L}\left(\frac{1}{\rho\sigma^2} + 1\right)^{1/2} \; ; \quad A_m^- = \left(\frac{m\pi}{L}\right)^2 - \frac{1}{\sqrt{2}}\rho\sigma\frac{m\pi}{L}\left(\frac{1}{\rho\sigma^2} + 1\right)^{1/2} ;$$

$$B_m = \frac{1}{\sqrt{2}}\rho\sigma\frac{m\pi}{L}\left(\frac{1}{\rho\sigma^2} - 1\right)^{1/2} .$$

On substituting Eq.(3-18) into Eq.(3-8), one obtains the displacement component u for the i^{th} panel:

$$\begin{aligned}
\bar{u}^i_{(\xi,\eta)} = \cos\frac{m\pi\xi}{L}\Bigg\{ & \sum_{j=1}^{R_{1m}}\left[A^i_j p^{(r)}_{mj} e^{-\alpha_{mj}\eta} + A^i_{j+R_{1m}} p^{(r)}_{mj} e^{\alpha_{mj}\eta}\right] \\
& + \sum_{j=1}^{R_{2m}}\left[A^i_{j+a_m} p^{(im)}_{mj}\cos\beta_{mj}\eta + A^i_{j+b_m} p^{(im)}_{mj}\sin\beta_{mj}\eta\right] \\
& + \sum_{j=1}^{R_{3m}}\Big[A^i_{j+c_m} e^{-\gamma_{mj}\eta}\left(p^{(c)}_{mj}\cos\delta_{mj}\eta + \bar{p}^{(c)}_{mj}\sin\delta_{mj}\eta\right) \\
& \quad + A^i_{j+d_m} e^{-\gamma_{mj}\eta}\left(p^{(c)}_{mj}\sin\delta_{mj}\eta - \bar{p}^{(c)}_{mj}\cos\delta_{mj}\eta\right) \\
& \quad + A^i_{j+e_m} e^{\gamma_{mj}\eta}\left(p^{(c)}_{mj}\cos\delta_{mj}\eta - \bar{p}^{(c)}_{mj}\sin\delta_{mj}\eta\right) \\
& \quad + A^i_{j+f_m} e^{\gamma_{mj}\eta}\left(p^{(c)}_{mj}\sin\delta_{mj}\eta + \bar{p}^{(c)}_{mj}\cos\delta_{mj}\eta\right)\Big]\Bigg\} ,
\end{aligned} \tag{3-19}$$

where

$$p_{mj}^{(r)} = \frac{1}{\rho^2}\left[\left(\frac{m\pi}{L}\right)^{-3}\alpha_{mj}^{6} + \left(\frac{m\pi}{L}\right)^{-1}(\nu-2)\alpha_{mj}^{4} + \left(\frac{m\pi}{L}\right)(1-2\nu)\alpha_{mj}^{2} + \left(\frac{m\pi}{L}\right)^{3}\right]$$

$$- 2\sigma^2\left[\left(\frac{m\pi}{L}\right)^{-1}\alpha_{mj}^{2} + \nu\left(\frac{m\pi}{L}\right)\right],$$

$$p_{mj}^{(im)} = \frac{1}{\rho^2}\left[-\left(\frac{m\pi}{L}\right)^{-3}\beta_{mj}^{6} + \left(\frac{m\pi}{L}\right)^{-1}(\nu-2)\beta_{mj}^{4} - \left(\frac{m\pi}{L}\right)(1-2\nu)\beta_{mj}^{2} + \left(\frac{m\pi}{L}\right)^{3}\right]$$

$$+ 2\sigma^2\left[\left(\frac{m\pi}{L}\right)^{-1}\beta_{mj}^{2} - \nu\left(\frac{m\pi}{L}\right)\right],$$

$$p_{mj}^{(c)} = \frac{1}{\rho^2}\left[\left(\frac{m\pi}{L}\right)^{-3}(\gamma_{mj}^{6} - 15\gamma_{mj}^{4}\delta_{mj}^{2} + 15\gamma_{mj}^{2}\delta_{mj}^{4} - \delta_{mj}^{6}) + \left(\frac{m\pi}{L}\right)^{-1}(\nu-2)(\gamma_{mj}^{4} - 6\gamma_{mj}^{2}\delta_{mj}^{2} + \delta_{mj}^{4})\right.$$

$$\left. + \left(\frac{m\pi}{L}\right)(1-2\nu)(\gamma_{mj}^{2} - \delta_{mj}^{2}) + \nu\left(\frac{m\pi}{L}\right)^{3}\right] - 2\sigma^2\left[\left(\frac{m\pi}{L}\right)^{-1}(\gamma_{mj}^{2} - \delta_{mj}^{2}) + \nu\left(\frac{m\pi}{L}\right)\right],$$

$$\bar{p}_{mj}^{(c)} = \frac{1}{\rho^2}\left[\left(\frac{m\pi}{L}\right)^{-3}(6\gamma_{mj}^{5}\delta_{mj} - 20\gamma_{mj}^{3}\delta_{mj}^{3} + 6\gamma_{mj}\delta_{mj}^{5}) + \left(\frac{m\pi}{L}\right)^{-1}(\nu-2)(4\gamma_{mj}^{3}\delta_{mj} - 4\gamma_{mj}\delta_{mj}^{3})\right.$$

$$\left. + \left(\frac{m\pi}{L}\right)(1-2\nu)(2\gamma_{mj}\delta_{mj})\right] - 2\sigma^2\left(\frac{m\pi}{L}\right)^{-1}(2\gamma_{mj}\delta_{mj}).$$

Similarly, substitution of Eq.(3-18) into Eq.(3-9) yields the displacement component v for the i^{th} panel:

$$\bar{v}^{i}_{(\xi,\eta)} = \sin\frac{m\pi\xi}{L}\left\{ \sum_{j=1}^{R_{1m}}\left[-A_j^{i}\, q_{mj}^{(r)}\, e^{-\alpha_{mj}\eta} + A_{j+R_{1m}}^{i}\, q_{mj}^{(r)}\, e^{\alpha_{mj}\eta}\right]\right. \qquad (3\text{-}20)$$

$$+ \sum_{j=1}^{R_{2m}}\left[-A_{j+a_m}^{i}\, q_{mj}^{(im)}\sin\beta_{mj}\eta + A_{j+b_m}^{i}\, q_{mj}^{(im)}\cos\beta_{mj}\eta\right]$$

$$+ \sum_{j=1}^{R_{3m}}\left[A_{j+c_m}^{i}\, e^{-\gamma_{mj}\eta}(-q_{mj}^{(c)}\cos\delta_{mj}\eta - \bar{q}_{mj}^{(c)}\sin\delta_{mj}\eta)\right.$$

$$+ A^i_{j+d_m} e^{-\gamma_{mj}\eta} (- q^{(c)}_{mj} \sin \delta_{mj}\eta + \bar{q}^{(c)}_{mj} \cos \delta_{mj}\eta)$$

$$+ A^i_{j+e_m} e^{\gamma_{mj}\eta} (q^{(c)}_{mj} \cos \delta_{mj}\eta - \bar{q}^{(c)}_{mj} \sin \delta_{mj}\eta)$$

$$+ A^i_{j+f_m} e^{\gamma_{mj}\eta} (q^{(c)}_{mj} \sin \delta_{mj}\eta + \bar{q}^{(c)}_{mj} \cos \delta_{mj}\eta)] \} ,$$

where

$$q^{(r)}_{mj} = - \frac{\alpha_{mj}}{\rho^2} [(\tfrac{m\pi}{L})^{-4} \alpha^6_{mj} - (4+\nu)(\tfrac{m\pi}{L})^{-2} \alpha^4_{mj} + (5+2\nu)\alpha^2_{mj} - (\tfrac{m\pi}{L})^2 (2+\nu)]$$

$$+ 2\sigma^2 \alpha_{mj} [(\tfrac{m\pi}{L})^{-2} \alpha^2_{mj} - (2+\nu)] ;$$

$$q^{(im)}_{mj} = \frac{\beta_{mj}}{\rho^2} [(\tfrac{m\pi}{L})^{-4} \beta^6_{mj} + (4+\nu)(\tfrac{m\pi}{L})^{-2} \beta^4_{mj} + (5+2\nu)\beta^2_{mj} + (\tfrac{m\pi}{L})^2 (2+\nu)]$$

$$- 2\sigma^2 \beta_{mj} [(\tfrac{m\pi}{L})^{-2} \beta^2_{mj} + (2+\nu)] ;$$

$$q^{(c)}_{mj} = - \frac{1}{\rho^2} [(\tfrac{m\pi}{L})^{-4} (\gamma^7_{mj} - 21\gamma^5_{mj}\delta^2_{mj} + 35\gamma^3_{mj}\delta^4_{mj} - 7\gamma_{mj}\delta^6_{mj}) - (\tfrac{m\pi}{L})^{-2} (4+\nu) (\gamma^5_{mj}$$

$$- 10\gamma^3_{mj}\delta^2_{mj} + 5\gamma_{mj}\delta^4_{mj}) + (5+2\nu)(\gamma^3_{mj} - 3\gamma_{mj}\delta^2_{mj}) - (\tfrac{m\pi}{L})^2 (2+\nu)\gamma_{mj}]$$

$$+ 2\sigma^2 [(\tfrac{m\pi}{L})^{-2} (\gamma^3_{mj} - 3\gamma_{mj}\delta^2_{mj}) - (2+\nu)\gamma_{mj}] ;$$

$$\bar{q}^{(c)}_{mj} = - \frac{1}{\rho^2} [(\tfrac{m\pi}{L})^{-4} (7\gamma^6_{mj} - 35\gamma^4_{mj}\delta^3_{mj} + 21\gamma^2_{mj}\delta^5_{mj} - \delta^7_{mj}) - (\tfrac{m\pi}{L})^{-2} (4+\nu) (5\gamma^4_{mj}\delta_{mj}$$

$$- 10\gamma^2_{mj}\delta^3_{mj} + \delta^5_{mj}) + (5+2\nu)(3\gamma^2_{mj}\delta_{mj} - \delta^3_{mj}) - (2+\nu)(\tfrac{m\pi}{L})^2 \delta_{mj}]$$

$$+ 2\sigma^2 [(\tfrac{m\pi}{L})^{-2} (3\gamma^2_{mj}\delta_{mj} - \delta^3_{mj}) + (2+\nu)\delta_{mj}] .$$

It is to be noted that the previous expressions for the displacement components u and v may also be obtained by the method of coefficient comparison as given in Appendix C. Numerical calculations made for several cases according to both procedures yield identical results. Some expressions for the derivatives of the functions $e^{\pm\alpha\xi}\cos\beta\xi$ and $e^{\pm\alpha\xi}\sin\beta\xi$, which are used in deriving the previous expressions, are given in Appendix D.

Substitution of Eqs.(3-18), (3-19) and (3-20) into Eqs.(2-40) through (2-46) results in the following expressions for stress resultants, stress couples and effective transverse shear stress resultants for the i^{th} panel:

$$\begin{aligned}\bar{N}^{i}_{x(\xi,\eta)} = \sin\frac{m\pi\xi}{L}\Big\{ &\sum_{j=1}^{R_{1m}}\big[A^{i}_{j}\, n^{(r)}_{xmj}\, e^{-\alpha_{mj}\eta} + A^{i}_{j+R_{1m}}\, n^{(r)}_{xmj}\, e^{\alpha_{mj}\eta}\big] \\ &+ \sum_{j=1}^{R_{2m}}\big[A^{i}_{j+a_m}\, n^{(im)}_{xmj}\cos\beta_{mj}\eta + A^{i}_{j+b_m}\, n^{(im)}_{xmj}\sin\beta_{mj}\eta\big] \\ &+ \sum_{j=1}^{R_{3m}}\big[A^{i}_{j+c_m}\, e^{-\delta_{mj}\eta}\,(n^{(c)}_{xmj}\cos\delta_{mj}\eta + \bar{n}^{(c)}_{xmj}\sin\delta_{mj}\eta) \\ &\quad + A^{i}_{j+d_m}\, e^{-\gamma_{mj}\eta}\,(n^{(c)}_{xmj}\sin\delta_{mj}\eta - \bar{n}^{(c)}_{xmj}\cos\delta_{mj}\eta) \\ &\quad + A^{i}_{j+e_m}\, e^{\gamma_{mj}\eta}\,(n^{(c)}_{xmj}\cos\delta_{mj}\eta - \bar{n}^{(c)}_{xmj}\sin\delta_{mj}\eta) \\ &\quad + A^{i}_{j+f_m}\, e^{\gamma_{mj}\eta}\,(n^{(c)}_{xmj}\sin\delta_{mj}\eta + \bar{n}^{(c)}_{xmj}\cos\delta_{mj}\eta)\big]\Big\}, \end{aligned} \tag{3-21}$$

where

$$n^{(r)}_{xmj} = \nu\,(q^{(r)}_{mj}\alpha_{mj} - 1) - \frac{m\pi}{L}\, p^{(r)}_{mj} \quad ;$$

$$\eta^{(im)}_{xmj} = \nu(q^{(im)}_{mj}\beta_{mj}+1) + \frac{m\pi}{L}p^{(im)}_{mj} ;$$

$$\eta^{(c)}_{xmj} = \nu(\gamma_{mj}q^{(c)}_{mj} - \delta_{mj}\bar{q}^{(c)}_{mj} - 1) - \frac{m\pi}{L}p^{(c)}_{mj} ;$$

$$\bar{\eta}^{(c)}_{xmj} = \nu(\gamma_{mj}\bar{q}^{(c)}_{mj} + \delta_{mj}q^{(c)}_{mj}) - \frac{m\pi}{L}\bar{p}^{(c)}_{mj} .$$

$$\begin{aligned}
\bar{N}^i_{y(\xi,\eta)} = \sin\frac{m\pi\xi}{L}\Big\{ & \sum_{j=1}^{R_{1m}}\left[A^i_j\eta^{(r)}_{ymj}e^{-\alpha_{mj}\eta} + A^i_{j+R_{1m}}\eta^{(r)}_{ymj}e^{\alpha_{mj}\eta}\right] \\
& + \sum_{j=1}^{R_{2m}}\left[-A^i_{j+a_m}\eta^{(im)}_{ymj}\cos\beta_{mj} - A^i_{j+b_m}\eta^{(im)}_{ymj}\sin\beta_{mj}\eta\right] \\
& + \sum_{j=1}^{R_{3m}}\Big[A^i_{j+c_m}e^{-\gamma_{mj}\eta}(\eta^{(c)}_{ymj}\cos\delta_{mj}\eta + \bar{\eta}^{(c)}_{ymj}\sin\delta_{mj}\eta) \\
& \quad + A^i_{j+d_m}e^{-\gamma_{mj}\eta}(\eta^{(c)}_{ymj}\sin\delta_{mj}\eta - \bar{\eta}^{(c)}_{ymj}\cos\delta_{mj}\eta) \\
& \quad + A^i_{j+e_m}e^{\gamma_{mj}\eta}(\eta^{(c)}_{ymj}\cos\delta_{mj}\eta - \bar{\eta}^{(c)}_{ymj}\sin\delta_{mj}\eta) \\
& \quad + A^i_{j+f_m}e^{\gamma_{mj}\eta}(\eta^{(c)}_{ymj}\sin\delta_{mj}\eta + \bar{\eta}^{(c)}_{ymj}\cos\delta_{mj}\eta)\Big]\Big\} ,
\end{aligned} \tag{3-22}$$

where

$$\eta^{(r)}_{ymj} = q^{(r)}_{mj}\alpha_{mj} - 1 - \nu\frac{m\pi}{L}p^{(r)}_{mj} ;$$

$$\eta^{(im)}_{ymj} = q^{(im)}_{mj}\beta_{mj} + 1 + \nu\frac{m\pi}{L}p^{(im)}_{mj} ;$$

$$\eta^{(c)}_{ymj} = \gamma_{mj}q^{(c)}_{mj} - \delta_{mj}\bar{q}^{(c)}_{mj} - 1 - \nu\frac{m\pi}{L}p^{(c)}_{mj} ;$$

$$\bar{\eta}^{(c)}_{ymj} = \gamma_{mj}\bar{q}^{(c)}_{mj} + \delta_{mj}q^{(c)}_{mj} - \nu\frac{m\pi}{L}\bar{p}^{(c)}_{mj} ;$$

$$\bar{N}^{i}_{xy(\xi,\eta)} = \cos\frac{m\pi\xi}{L}\Big\{\sum_{j=1}^{R_{1m}}[-A^{i}_{j}\eta^{(r)}_{xymj}\, e^{-\alpha_{mj}\eta} + A^{i}_{j+R_{1m}}\eta^{(r)}_{xymj}\, e^{\alpha_{mj}\eta}] \tag{3-23}$$

$$+ \sum_{j=1}^{R_{2m}}[-A^{i}_{j+a_m}\eta^{(im)}_{xymj}\sin\beta_{mj}\eta + A^{i}_{j+b_m}\eta^{(im)}_{xymj}\cos\beta_{mj}\eta]$$

$$+ \sum_{j=1}^{R_{3m}}[A^{i}_{j+c_m}e^{-\gamma_{mj}\eta}(-\eta^{(c)}_{xymj}\cos\delta_{mj}\eta - \bar{\eta}^{(c)}_{xymj}\sin\delta_{mj}\eta)$$

$$+ A^{i}_{j+d_m}e^{-\gamma_{mj}\eta}(-\eta^{(c)}_{xymj}\sin\delta_{mj}\eta + \bar{\eta}^{(c)}_{xymj}\cos\delta_{mj}\eta)$$

$$+ A^{i}_{j+e_m}e^{\gamma_{mj}\eta}(\eta^{(c)}_{xymj}\cos\delta_{mj}\eta - \bar{\eta}^{(c)}_{xymj}\sin\delta_{mj}\eta)$$

$$+ A^{i}_{j+f_m}e^{\gamma_{mj}\eta}(\eta^{(c)}_{xymj}\sin\delta_{mj}\eta + \bar{\eta}^{(c)}_{xymj}\cos\delta_{mj}\eta)]\Big\},$$

where

$$\eta^{(r)}_{xymj} = \tfrac{1}{2}(1-\nu)(p^{(r)}_{mj}\alpha_{mj} + \tfrac{m\pi}{L}q^{(r)}_{mj}) \;;$$

$$\eta^{(im)}_{xymj} = \tfrac{1}{2}(1-\nu)(p^{(im)}_{mj}\beta_{mj} + \tfrac{m\pi}{L}q^{(im)}_{mj}) \;;$$

$$\eta^{(c)}_{xymj} = \tfrac{1}{2}(1-\nu)(p^{(c)}_{mj}\gamma_{mj} - \bar{p}^{(c)}_{mj}\delta_{mj} + \tfrac{m\pi}{L}q^{(c)}_{mj}) \;;$$

$$\bar{\eta}^{(c)}_{xymj} = \tfrac{1}{2}(1-\nu)(\bar{p}^{(c)}_{mj}\gamma_{mj} + p^{(c)}_{mj}\delta_{mj} + \tfrac{m\pi}{L}\bar{q}^{(c)}_{mj}) \;;$$

$$\bar{M}^{i}_{x(\xi,\eta)} = \sin\frac{m\pi\xi}{L}\Big\{\sum_{j=1}^{R_{1m}}[A^{i}_{j}M^{(r)}_{xmj}e^{-\alpha_{mj}\eta} + A^{i}_{j+R_{1m}}M^{(r)}_{xmj}e^{\alpha_{mj}\eta}] \tag{3-24}$$

$$+ \sum_{j=1}^{R_{2m}}[A^{i}_{j+a_m}M^{(im)}_{xmj}\cos\beta_{mj}\eta + A^{i}_{j+b_m}M^{(im)}_{xmj}\sin\beta_{mj}\eta]$$

$$+ \sum_{j=1}^{R_{3m}} \Big[A^i_{j+c_m} e^{-\gamma_{mj}\eta} (\mathfrak{M}^{(c)}_{xmj} \cos \delta_{mj}\eta + \bar{\mathfrak{M}}^{(c)}_{xmj} \sin \delta_{mj}\eta)$$

$$+ A^i_{j+d_m} e^{-\gamma_{mj}\eta} (\mathfrak{M}^{(c)}_{xmj} \sin \delta_{mj}\eta - \bar{\mathfrak{M}}^{(c)}_{xmj} \cos \delta_{mj}\eta)$$

$$+ A^i_{j+e_m} e^{\gamma_{mj}\eta} (\mathfrak{M}^{(c)}_{xmj} \cos \delta_{mj}\eta - \bar{\mathfrak{M}}^{(c)}_{xmj} \sin \delta_{mj}\eta)$$

$$+ A^i_{j+f_m} e^{\gamma_{mj}\eta} (\mathfrak{M}^{(c)}_{xmj} \sin \delta_{mj}\eta + \bar{\mathfrak{M}}^{(c)}_{xmj} \cos \delta_{mj}\eta) \Big] \Big\} ,$$

where

$$\mathfrak{M}^{(r)}_{xmj} = (\tfrac{m\pi}{L})^2 - \nu \alpha^2_{mj} \quad ; \quad \mathfrak{M}^{(im)}_{xmj} = (\tfrac{m\pi}{L})^2 + \nu \beta^2_{mj} \quad ;$$

$$\mathfrak{M}^{(c)}_{xmj} = (\tfrac{m\pi}{L})^2 - \nu (\gamma^2_{mj} - \delta^2_{mj}) \; ; \quad \bar{\mathfrak{M}}^{(c)}_{xmj} = -2\nu \gamma_{mj} \delta_{mj} \quad .$$

$$\bar{M}^i_{y(\xi,\eta)} = \sin \tfrac{m\pi}{L} \Big\{ \sum_{j=1}^{R_{1m}} \Big[A^i_j \mathfrak{M}^{(r)}_{ymj} e^{-\alpha_{mj}\eta} + A^i_{j+R_{1m}} \mathfrak{M}^{(r)}_{ymj} e^{\alpha_{mj}\eta} \Big] \tag{3-25}$$

$$+ \sum_{j=1}^{R_{2m}} \Big[A^i_{j+a_m} \mathfrak{M}^{(im)}_{ymj} \cos \beta_{mj}\eta + A^i_{j+b_m} \mathfrak{M}^{(im)}_{ymj} \sin \delta_{mj}\eta \Big]$$

$$+ \sum_{j=1}^{R_{3m}} \Big[A^i_{j+c_m} e^{-\gamma_{mj}\eta} (\mathfrak{M}^{(c)}_{ymj} \cos \delta_{mj}\eta + \bar{\mathfrak{M}}^{(c)}_{ymj} \sin \delta_{mj}\eta)$$

$$+ A^i_{j+d_m} e^{-\gamma_{mj}\eta} (\mathfrak{M}^{(c)}_{ymj} \sin \delta_{mj}\eta - \bar{\mathfrak{M}}^{(c)}_{ymj} \cos \delta_{mj}\eta)$$

$$+ A^i_{j+e_m} e^{\gamma_{mj}\eta} (\mathfrak{M}^{(c)}_{ymj} \cos \delta_{mj}\eta - \bar{\mathfrak{M}}^{(c)}_{ymj} \sin \delta_{mj}\eta)$$

$$+ A^i_{j+f_m} e^{\gamma_{mj}\eta} (\mathfrak{M}^{(c)}_{ymj} \sin \delta_{mj}\eta + \bar{\mathfrak{M}}^{(c)}_{ymj} \cos \delta_{mj}\eta) \Big] \Big\} ,$$

where

$$\mathcal{M}^{(r)}_{ymj} = \nu\left(\frac{m\pi}{L}\right)^2 - \alpha^2_{mj} \quad ; \quad \mathcal{M}^{(im)}_{ymj} = \nu\left(\frac{m\pi}{L}\right)^2 + \beta^2_{mj} \quad ;$$

$$\mathcal{M}^{(c)}_{ymj} = \nu\left(\frac{m\pi}{L}\right)^2 - (\gamma^2_{mj} - \delta^2_{mj}) \quad ; \quad \bar{\mathcal{M}}^{(c)}_{ymj} = -2\,\gamma_{mj}\,\delta_{mj}\,.$$

$$\bar{M}^{i}_{xy}(\xi,\eta) = -(1-\nu)\frac{m\pi}{L}\cos\frac{m\pi}{L}\Big\{\sum_{j=1}^{R_{1m}}\left[-A^{i}_{j}\alpha_{mj}e^{-\alpha_{mj}\eta} + A^{i}_{j+R_{1m}}e^{\alpha_{mj}\eta}\right] \tag{3-26}$$

$$+ \sum_{j=1}^{R_{2m}}\left[-A^{i}_{j+a_m}\beta_{mj}\sin\beta_{mj}\eta + A^{i}_{j+b_m}\beta_{mj}\cos\beta_{mj}\eta\right]$$

$$+ \sum_{j=1}^{R_{3m}}\Big[A^{i}_{j+c_m}e^{-\gamma_{mj}\eta}(-\gamma_{mj}\cos\delta_{mj}\eta - \delta_{mj}\sin\delta_{mj}\eta)$$

$$+ A^{i}_{j+d_m}e^{-\gamma_{mj}\eta}(-\gamma_{mj}\sin\delta_{mj}\eta + \delta_{mj}\cos\delta_{mj}\eta)$$

$$+ A^{i}_{j+e_m}e^{\gamma_{mj}\eta}(\gamma_{mj}\cos\delta_{mj}\eta - \delta_{mj}\sin\delta_{mj}\eta)$$

$$+ A^{i}_{j+f_m}e^{\gamma_{mj}\eta}(\gamma_{mj}\sin\delta_{mj}\eta + \delta_{mj}\cos\delta_{mj}\eta))\Big]\Big\},$$

$$\bar{V}^{i}_{x}(\xi,\eta) = \cos\frac{m\pi\xi}{L}\Big\{\sum_{j=1}^{R_{1m}}\left[A^{i}_{j}Q^{(r)}_{xmj}e^{-\alpha_{mj}\eta} + A^{i}_{j+R_{1m}}Q^{(r)}_{xmj}e^{\alpha_{mj}\eta}\right] \tag{3-27}$$

$$+ \sum_{j=1}^{R_{2m}}\left[A^{i}_{j+a_m}Q^{(im)}_{xmj}\cos\beta_{mj}\eta + A^{i}_{j+b_m}Q^{(im)}_{xmj}\sin\beta_{mj}\eta\right]$$

$$+ \sum_{j=1}^{R_{3m}}\Big[A^{i}_{j+c_m}e^{-\gamma_{mj}\eta}(Q^{(c)}_{xmj}\cos\delta_{mj}\eta + \bar{Q}^{(c)}_{xmj}\sin\delta_{mj}\eta)$$

$$+ A^{i}_{j+d_m}e^{-\gamma_{mj}\eta}(Q^{(c)}_{xmj}\sin\delta_{mj}\eta - \bar{Q}^{(c)}_{xmj}\cos\delta_{mj}\eta)$$

$$+ A^{i}_{j+e_m} e^{\gamma_{mj}\eta} (Q^{(c)}_{xmj} \cos\delta_{mj}\eta - \bar{Q}^{(c)}_{xmj} \sin\delta_{mj}\eta)$$

$$+ A^{i}_{j+f_m} e^{\gamma_{mj}\eta} (Q^{(c)}_{xmj} \sin\delta_{mj}\eta + \bar{Q}^{(c)}_{xmj} \cos\delta_{mj}\eta)] \Big\} ,$$

where

$$Q^{(r)}_{xmj} = (\tfrac{m\pi}{L})^3 - (2-\nu)\tfrac{m\pi}{L} \alpha^2_{mj} \quad ;$$

$$Q^{(im)}_{xmj} = (\tfrac{m\pi}{L})^3 + (2-\nu)\tfrac{m\pi}{L} \beta^2_{mj} \quad ;$$

$$Q^{(c)}_{xmj} = (\tfrac{m\pi}{L})^3 - (2-\nu)\tfrac{m\pi}{L} (\gamma^2_{mj} - \delta^2_{mj}) \quad ;$$

$$\bar{Q}^{(c)}_{xmj} = -2(2-\nu)\tfrac{m\pi}{L} \gamma_{mj}\delta_{mj} \quad .$$

$$\bar{V}^{i}_{y(\xi,\eta)} = \sin\frac{m\pi\xi}{L} \Big\{ \sum_{j=1}^{R_{1m}} [-A^{i}_{j} Q^{(r)}_{ymj} e^{-\alpha_{mj}\eta} + A^{i}_{j+R_{1m}} Q^{(r)}_{ymj} e^{\alpha_{mj}\eta}] \qquad (3\text{-}28)$$

$$+ \sum_{j=1}^{R_{2m}} [-A^{i}_{j+a_m} Q^{(im)}_{ymj} \sin\beta_{mj} + A^{i}_{j+b_m} Q^{(im)}_{ymj} \cos\beta_{mj}\eta]$$

$$+ \sum_{j=1}^{R_{3m}} [A^{i}_{j+c_m} e^{-\gamma_{mj}\eta} (-Q^{(c)}_{ymj} \cos\delta_{mj}\eta + \bar{Q}^{(c)}_{ymj} \sin\delta_{mj}\eta)$$

$$+ A^{i}_{j+d_m} e^{-\gamma_{mj}\eta} (-Q^{(c)}_{ymj} \sin\delta_{mj}\eta - \bar{Q}^{(c)}_{ymj} \cos\delta_{mj}\eta)$$

$$+ A^{i}_{j+e_m} e^{\gamma_{mj}\eta} (Q^{(c)}_{ymj} \cos\delta_{mj}\eta + \bar{Q}^{(c)}_{ymj} \sin\delta_{mj}\eta)$$

$$+ A^{i}_{j+f_m} e^{\gamma_{mj}\eta} (Q^{(c)}_{ymj} \sin\delta_{mj}\eta - \bar{Q}^{(c)}_{ymj} \cos\delta_{mj}\eta)] \Big\} ,$$

where

$$Q^{(r)}_{ymj} = (2-\nu)\left(\frac{m\pi}{L}\right)^2 \alpha_{mj} - \alpha^3_{mj} \quad ;$$

$$Q^{(im)}_{ymj} = (2-\nu)\left(\frac{m\pi}{L}\right)^2 \beta_{mj} + \beta^3_{mj} \quad ;$$

$$Q^{(c)}_{ymj} = (2-\nu)\left(\frac{m\pi}{L}\right)^2 \gamma_{mj} - \left(\gamma^3_{mj} - 3\,\gamma_{mj}\,\delta^2_{mj}\right) \; ;$$

$$\bar{Q}^{(c)}_{ymj} = -(2-\nu)\left(\frac{m\pi}{L}\right)^2 \delta_{mj} + \left(3\gamma^2_{mj}\delta_{mj} - \delta^3_{mj}\right) \; .$$

Substituting the previous expressions, Eqs.(3-18) through (3-28), for the displacement components as well as the stress resultants and moments into the continuity and compatibility conditions, Eqs.(2-48) through (2-55), and then putting all terms on the left hand side, one may get the trigonometric functions as

$$F_{m(\sigma)} \sin\frac{m\pi\xi}{L} = 0 \qquad \text{or} \qquad F_{m(\sigma)} \cos\frac{m\pi\xi}{L} = 0 \quad ,$$

which must be held for any value of ξ. Thus the coefficients of each trigonometric function must be zero and, for each m, one obtains the algebraic equations, such as

$$\sum_{j=1}^{8} A_j^{i-1}\, \Phi_{mkj}(\bar\eta_i) + \sum_{j=1}^{8} A_j^{i}\, \bar\Psi_{mkj}(\eta_i) = 0 \qquad (3\text{-}29)$$

for $k = 1,2\ldots8$, and $i = 1,2\ldots N$

where $\bar\eta_i = \bar\eta$ for $i \neq 1$ and $\bar\eta_i = 2\pi$ for $i \neq 1$, and N is the total number of stringers;

$$\Phi_{m1j} = -e^{-\alpha_{mj}\bar{\eta}_i} \quad ; \quad \Phi_{m1j+\ell_{1m}} = -e^{\alpha_{mj}\bar{\eta}_i} \quad ; \tag{3-30a}$$

$$\Phi_{m1j+a_m} = -\cos\beta_{mj}\bar{\eta}_i \quad ; \quad \Phi_{m1j+b_m} = -\sin\beta_{mj}\bar{\eta}_i \quad ; \tag{3-30b}$$

$$\Phi_{m1j+c_m} = -e^{-\gamma_{mj}\bar{\eta}_i}\cos\delta_{mj}\bar{\eta}_i \quad ; \quad \Phi_{m1j+d_m} = -e^{-\gamma_{mj}\bar{\eta}_i}\sin\delta_{mj}\bar{\eta}_i \quad ; \tag{3-30c}$$

$$\Phi_{m1j+e_m} = -e^{\gamma_{mj}\bar{\eta}_i}\cos\delta_{mj}\bar{\eta}_i \quad ; \quad \Phi_{m1j+f_m} = -e^{\gamma_{mj}\bar{\eta}_i}\sin\delta_{mj}\bar{\eta}_i \quad ;$$

$$\bar{\Psi}_{m1j} = e^{-\alpha_{mj}\eta_i} \quad ; \quad \bar{\Psi}_{m1j+\ell_{1m}} = e^{\alpha_{mj}\eta_i} \quad ; \tag{3-31a}$$

$$\bar{\Psi}_{m1j+a_m} = \cos\beta_{mj}\eta_i \quad ; \quad \bar{\Psi}_{m1j+b_m} = \sin\beta_{mj}\eta_i \quad ; \tag{3-31b}$$

$$\bar{\Psi}_{m1j+c_m} = e^{-\delta_{mj}\eta_i}\cos\delta_{mj}\eta_i \quad ; \quad \bar{\Psi}_{m1j+d_m} = e^{-\gamma_{mj}\eta_i}\sin\delta_{mj}\eta_i \tag{3-31c}$$

$$\bar{\Psi}_{m1j+e_m} = e^{\gamma_{mj}\eta_i}\cos\delta_{mj}\eta_i \quad ; \quad \bar{\Psi}_{m1j+f_m} = e^{\delta_{mj}\eta_i}\sin\delta_{mj}\eta_i \quad ;$$

$$\Phi_{m2j} = -p_{mj}^{(r)} e^{-\alpha_{mj}\bar{\eta}_i} \quad ; \quad \Phi_{m2j+\ell_{1m}} = -p_{mj}^{(r)} e^{\alpha_{mj}\bar{\eta}_i} \quad ; \tag{3-32a}$$

$$\Phi_{m2j+a_m} = -p_{mj}^{(im)}\cos\beta_{mj}\bar{\eta}_i \quad ; \quad \Phi_{m2j+b_m} = -p_{mj}^{(im)}\sin\beta_{mj}\bar{\eta}_i \quad ; \tag{3-32b}$$

$$\begin{Bmatrix} \Phi_{m2j+c_m} \\ \\ \Phi_{m2j+d_m} \end{Bmatrix} = -e^{-\gamma_{mj}\bar{\eta}_i}\left[p_{mj}^{(c)} \begin{Bmatrix} \cos\delta_{mj}\bar{\eta}_i \\ \\ \sin\delta_{mj}\bar{\eta}_i \end{Bmatrix} + \bar{p}_{mj}^{(c)} \begin{Bmatrix} \sin\delta_{mj}\bar{\eta}_i \\ \\ -\cos\delta_{mj}\bar{\eta}_i \end{Bmatrix} \right] \tag{3-32c}$$

$$\begin{Bmatrix} \phi_{m2,j+e_m} \\ \phi_{m2,j+f_m} \end{Bmatrix} = -e^{\gamma_{mj}\bar{\eta}_i} \left[p_{mj}^{(c)} \begin{Bmatrix} \cos\delta_{mj}\bar{\eta}_i \\ \sin\delta_{mj}\bar{\eta}_i \end{Bmatrix} + \bar{p}_{mj}^{(c)} \begin{Bmatrix} -\sin\delta_{mj}\bar{\eta}_i \\ \cos\delta_{mj}\bar{\eta}_i \end{Bmatrix} \right]$$

$$\bar{\Psi}_{m,2j} = p_{mj}^{(r)} e^{-\alpha_{mj}\eta_i} \quad ; \quad \bar{\Psi}_{m2,j+k_m} = p_{mj}^{(r)} e^{\alpha_{mj}\eta_i} \quad ; \tag{3-33a}$$

$$\bar{\Psi}_{m,2,j+a_m} = p_{mj}^{(im)} \cos\beta_{mj}\eta_i \; ; \; \bar{\Psi}_{m2j+b_m} = p_{mj}^{(im)} \sin\beta_{mj}\eta_i \; ; \tag{3-33b}$$

$$\begin{Bmatrix} \bar{\Psi}_{m2j+c_m} \\ \bar{\Psi}_{m2j+d_m} \end{Bmatrix} = e^{-\gamma_{mj}\eta_i} \left[p_{mj}^{(c)} \begin{Bmatrix} \cos\delta_{mj}\eta_i \\ \sin\delta_{mj}\eta_i \end{Bmatrix} + \bar{p}_{mj}^{(c)} \begin{Bmatrix} \sin\delta_{mj}\eta_i \\ -\cos\delta_{mj}\eta_i \end{Bmatrix} \right] \tag{3-33c}$$

$$\begin{Bmatrix} \bar{\Psi}_{m2j+e_m} \\ \bar{\Psi}_{m2j+f_m} \end{Bmatrix} = e^{\gamma_{mj}\eta_i} \left[p_{mj}^{(c)} \begin{Bmatrix} \cos\delta_{mj}\eta_i \\ \sin\delta_{mj}\eta_i \end{Bmatrix} + \bar{p}_{mj}^{(c)} \begin{Bmatrix} -\sin\delta_{mj}\eta_i \\ \cos\delta_{mj}\eta_i \end{Bmatrix} \right]$$

$$\phi_{m3j} = -q_{mj}^{(r)} e^{-\alpha_{mj}\bar{\eta}_i} ; \quad \phi_{m3j+k_m} = -q_{mj}^{(r)} e^{\alpha_{mj}\bar{\eta}_i} \quad ; \tag{3-34a}$$

$$\phi_{m3j+a_m} = q_{mj}^{(im)} \sin\beta_{mj}\bar{\eta}_i \; ; \quad \phi_{m3j+b_m} = -q_{mj}^{(im)} \cos\beta_{mj}\bar{\eta}_i \; ; \tag{3-34b}$$

$$\begin{Bmatrix} \phi_{m3j+c_m} \\ \phi_{m3j+d_m} \end{Bmatrix} = e^{-\gamma_{mj}\bar{\eta}_i} \left[q_{mj}^{(c)} \begin{Bmatrix} \cos\delta_{mj}\bar{\eta}_i \\ \sin\delta_{mj}\bar{\eta}_i \end{Bmatrix} + \bar{q}_{mj}^{(c)} \begin{Bmatrix} \sin\delta_{mj}\bar{\eta}_i \\ -\cos\delta_{mj}\bar{\eta}_i \end{Bmatrix} \right] \tag{3-34c}$$

$$\begin{Bmatrix} \phi_{m3j+e_m} \\ \phi_{m3j+f_m} \end{Bmatrix} = -e^{\gamma_{mj}\bar{\eta}_i} \left[q_{mj}^{(c)} \begin{Bmatrix} \cos\delta_{mj}\bar{\eta}_i \\ \sin\delta_{mj}\bar{\eta}_i \end{Bmatrix} + \bar{q}_{mj}^{(c)} \begin{Bmatrix} -\sin\delta_{mj}\bar{\eta}_i \\ \cos\delta_{mj}\bar{\eta}_i \end{Bmatrix} \right]$$

$$\bar{\Psi}_{m3j} = q^{(r)}_{mj}\, e^{-\alpha_{mj}\eta_i} \; ; \qquad \bar{\Psi}_{m3j+R_{im}} = q^{(r)}_{mj}\, e^{\alpha_{mj}\eta_i} \; ; \tag{3-35a}$$

$$\bar{\Psi}_{m3j+a_m} = -q^{(im)}_{mj} \sin\beta_{mj}\eta_i \; ; \; \bar{\Psi}_{m3j+b_m} = q^{(im)}_{mj} \cos\beta_{mj}\eta_i \; ; \tag{3-35b}$$

$$\begin{Bmatrix} \bar{\Psi}_{m3j+c_m} \\ \\ \bar{\Psi}_{m3j+d_m} \end{Bmatrix} = e^{-\gamma_{mj}\eta_i}\left[-q^{(c)}_{mj} \begin{Bmatrix} \cos\delta_{mj}\eta_i \\ \\ \sin\delta_{mj}\eta_i \end{Bmatrix} + \bar{q}^{(c)}_{mj} \begin{Bmatrix} -\sin\delta_{mj}\eta_i \\ \\ \cos\delta_{mj}\eta_i \end{Bmatrix} \right] \tag{3-35c}$$

$$\begin{Bmatrix} \bar{\Psi}_{m3j+e_m} \\ \\ \bar{\Psi}_{m3j+f_m} \end{Bmatrix} = e^{\gamma_{mj}\eta_i}\left[q^{(c)}_{mj} \begin{Bmatrix} \cos\delta_{mj}\eta_i \\ \\ \sin\delta_{mj}\eta_i \end{Bmatrix} + \bar{q}^{(c)}_{mj} \begin{Bmatrix} -\sin\delta_{mj}\eta_i \\ \\ \cos\delta_{mj}\eta_i \end{Bmatrix} \right]$$

$$\phi_{m4j} = \alpha_{mj}\, e^{-\alpha_{mj}\bar{\eta}_i} \; ; \quad \phi_{m4j+R_{im}} = -\alpha_{mj}\, e^{\alpha_{mj}\bar{\eta}_i} \; ; \tag{3-36a}$$

$$\phi_{m4j+a_m} = \beta_{mj} \sin\beta_{mj}\bar{\eta}_i \; ; \quad \phi_{m4j+b_m} = -\beta_{mj} \cos\beta_{mj}\bar{\eta}_i \; ; \tag{3-36b}$$

$$\begin{Bmatrix} \phi_{m4j+c_m} \\ \\ \phi_{m4j+d_m} \end{Bmatrix} = e^{-\gamma_{mj}\bar{\eta}_i}\left[\gamma_{mj} \begin{Bmatrix} \cos\delta_{mj}\bar{\eta}_i \\ \\ \sin\delta_{mj}\bar{\eta}_i \end{Bmatrix} + \delta_{mj} \begin{Bmatrix} \sin\delta_{mj}\bar{\eta}_i \\ \\ -\cos\delta_{mj}\bar{\eta}_i \end{Bmatrix} \right] \tag{3-36c}$$

$$\begin{Bmatrix} \phi_{m4j+e_m} \\ \\ \phi_{m4j+f_m} \end{Bmatrix} = e^{\gamma_{mj}\bar{\eta}_i}\left[-\gamma_{mj} \begin{Bmatrix} \cos\delta_{mj}\bar{\eta}_i \\ \\ \sin\delta_{mj}\bar{\eta}_i \end{Bmatrix} + \delta_{mj} \begin{Bmatrix} \sin\delta_{mj}\bar{\eta}_i \\ \\ -\cos\delta_{mj}\bar{\eta}_i \end{Bmatrix} \right]$$

$$\bar{\Psi}_{m4j} = -\alpha_{mj}\, e^{-\alpha_{mj}\eta_i} \; ; \quad \bar{\Psi}_{m4j+R_{im}} = \alpha_{mj}\, e^{\alpha_{mj}\eta_i} \; ; \tag{3-37a}$$

$$\bar{\Psi}_{m4j+a_m} = -\beta_{mj} \sin\beta_{mj}\eta_i \; ; \; \bar{\Psi}_{m4j+b_m} = \beta_{mj} \cos\beta_{mj}\eta_i \; ; \tag{3-37b}$$

$$\begin{Bmatrix} \bar{\Psi}_{m4j+c_m} \\ \\ \bar{\Psi}_{m4j+d_m} \end{Bmatrix} = e^{-\gamma_{mj}\eta_i}\left[-\gamma_{mj}\begin{Bmatrix} \cos\delta_{mj}\eta_i \\ \\ \sin\delta_{mj}\eta_i \end{Bmatrix} + \delta_{mj}\begin{Bmatrix} -\sin\delta_{mj}\eta_i \\ \\ \cos\delta_{mj}\eta_i \end{Bmatrix} \right] \tag{3-37c}$$

$$\begin{Bmatrix} \bar{\Psi}_{m4j+e_m} \\ \\ \bar{\Psi}_{m4j+f_m} \end{Bmatrix} = e^{\gamma_{mj}\eta_i}\left[\gamma_{mj}\begin{Bmatrix} \cos\delta_{mj}\eta_i \\ \\ \sin\delta_{mj}\eta_i \end{Bmatrix} + \delta_{mj}\begin{Bmatrix} -\sin\delta_{mj}\eta_i \\ \\ \cos\delta_{mj}\eta_i \end{Bmatrix} \right]$$

$$\begin{Bmatrix} \Phi_{m5j} \\ \\ \Phi_{m5j+k_m} \end{Bmatrix} = \left[Q^{(r)}_{ymj} - k_3\bar{h}_z\left(\frac{m\pi}{L}\right)\eta^{(r)}_{xymj} \right]\begin{Bmatrix} -e^{-\alpha_{mj}\bar{\eta}_i} \\ \\ e^{\alpha_{mj}\bar{\eta}_i} \end{Bmatrix} \tag{3-38a}$$

$$\begin{Bmatrix} \Phi_{m5j+a_m} \\ \\ \Phi_{m5j+b_m} \end{Bmatrix} = \left[Q^{(im)}_{ymj} - k_3\bar{h}_z\left(\frac{m\pi}{L}\right)\eta^{(im)}_{xymj} \right]\begin{Bmatrix} -\sin\beta_{mj}\bar{\eta}_i \\ \\ \cos\beta_{mj}\bar{\eta}_i \end{Bmatrix} \tag{3-38b}$$

$$\begin{Bmatrix} \Phi_{m5j+c_m} \\ \\ \Phi_{m5j+d_m} \end{Bmatrix} = e^{-\gamma_{mj}\bar{\eta}_i}\left[-\left[Q^{(c)}_{ymj} - k_3\bar{h}_z\left(\frac{m\pi}{L}\right)\eta^{(c)}_{xymj} \right]\begin{Bmatrix} \cos\delta_{mj}\bar{\eta}_i \\ \\ \sin\delta_{mj}\bar{\eta}_i \end{Bmatrix} \right. \tag{3-38c}$$

$$\left. + \left[\bar{Q}^{(c)}_{ymj} + k_3\bar{h}_z\left(\frac{m\pi}{L}\right)\bar{\eta}^{(c)}_{xymj} \right]\begin{Bmatrix} \sin\delta_{mj}\bar{\eta}_i \\ \\ -\cos\delta_{mj}\bar{\eta}_i \end{Bmatrix} \right]$$

$$\begin{Bmatrix} \Phi_{m5j+e_m} \\ \\ \Phi_{m5j+f_m} \end{Bmatrix} = e^{\gamma_{mj}\bar{\eta}_i}\left[\left[Q^{(c)}_{ymj} - k_3\bar{h}_z\left(\frac{m\pi}{L}\right)\eta^{(c)}_{xymj} \right]\begin{Bmatrix} \cos\delta_{mj}\bar{\eta}_i \\ \\ \sin\delta_{mj}\bar{\eta}_i \end{Bmatrix} \right.$$

$$\left. + \left[\bar{Q}^{(c)}_{ymj} + k_3\bar{h}_z\left(\frac{m\pi}{L}\right)\bar{\eta}^{(c)}_{xymj} \right]\begin{Bmatrix} \sin\delta_{mj}\bar{\eta}_i \\ \\ -\cos\delta_{mj}\bar{\eta}_i \end{Bmatrix} \right]$$

$$\bar{\Psi}_{m5j} = \Big[\bar{I}_y^i\left(\tfrac{m\pi}{L}\right)^4 - \bar{N}_o^i\left(\tfrac{m\pi}{L}\right)^2 - \Big(\bar{d}_1^i\bar{I}_y^i\left(\tfrac{m\pi}{L}\right)^4\alpha_{mj} + \bar{N}_o^i\bar{h}_y^i\left(\tfrac{m\pi}{L}\right)^2\alpha_{mj} + K_1^i\bar{I}_y^i\left(\tfrac{m\pi}{L}\right)^4 q_{mj}^{(r)} \qquad (3\text{-}39a)$$

$$- Q_{ymj}^{(r)} + K_3\bar{h}_z^i\left(\tfrac{m\pi}{L}\right) \eta_{xymj}^{(r)}\Big)\Big] e^{-\alpha_{mj}\eta_i} \;;$$

$$\bar{\Psi}_{m5j+k_{1m}} = \Big[\bar{I}_y^i\left(\tfrac{m\pi}{L}\right)^4 - \bar{N}_o^i\left(\tfrac{m\pi}{L}\right)^2 + \bar{d}_1^i\bar{I}_y^i\left(\tfrac{m\pi}{L}\right)^4\alpha_{mj} + \bar{N}_o^i\bar{h}_y^i\left(\tfrac{m\pi}{L}\right)^2\alpha_{mj} + K_1^i\bar{I}_y^i\left(\tfrac{m\pi}{L}\right)^4 q_{mj}^{(r)}$$

$$- Q_{ymj}^{(r)} + K_3\bar{h}_z^i\left(\tfrac{m\pi}{L}\right)\eta_{xymj}^{(r)}\Big] e^{\alpha_{mj}\eta_i} \;;$$

$$\begin{Bmatrix} \bar{\Psi}_{m5j+a_m} \\ \\ \bar{\Psi}_{m5j+b_m} \end{Bmatrix} = \Big[\bar{I}_y^i\left(\tfrac{m\pi}{L}\right)^4 - \bar{N}_o^i\left(\tfrac{m\pi}{L}\right)^2\Big] \begin{Bmatrix} \cos\beta_{mj}\eta_i \\ \\ \sin\beta_{mj}\eta_i \end{Bmatrix} \qquad (3\text{-}39b)$$

$$+ \Big[\bar{d}_1^i\bar{I}_y^i\left(\tfrac{m\pi}{L}\right)^4\beta_{mj} + \bar{N}_o^i\bar{h}_y^i\left(\tfrac{m\pi}{L}\right)^2\beta_{mj} + K_1^i\bar{I}_y^i\left(\tfrac{m\pi}{L}\right)^4 q_{mj}^{(im)} - Q_{ymj}^{(im)} + K_3\bar{h}_z^i\left(\tfrac{m\pi}{L}\right)\eta_{xymj}^{(im)}\Big] \begin{Bmatrix} -\sin\beta_{mj}\eta_i \\ \\ \cos\beta_{mj}\eta_i \end{Bmatrix}$$

$$\begin{Bmatrix} \bar{\Psi}_{m5j+c_m} \\ \\ \bar{\Psi}_{m5j+d_m} \end{Bmatrix} = e^{-\gamma_{mj}\eta_i}\left[\begin{bmatrix} \bar{I}_y^i\left(\tfrac{m\pi}{L}\right)^4 - \bar{N}_o^i\left(\tfrac{m\pi}{L}\right)^2 - \left(\bar{d}_1^i\bar{I}_y^i\left(\tfrac{m\pi}{L}\right)^4 + \bar{N}_o^i\bar{h}_y^i\left(\tfrac{m\pi}{L}\right)^2\right)\gamma_{mj} \\ \\ -K_1^i\bar{I}_y^i\left(\tfrac{m\pi}{L}\right)^4 q_{mj}^{(c)} + Q_{ymj}^{(c)} - K_3\bar{h}_z^i\left(\tfrac{m\pi}{L}\right)\eta_{xymj}^{(c)} \end{bmatrix} \begin{Bmatrix} \cos\delta_{mj}\eta_i \\ \\ \sin\delta_{mj}\eta_i \end{Bmatrix}\right. \qquad (3\text{-}39c)$$

$$\left. + \begin{bmatrix} \left(\bar{d}_1^i\bar{I}_y^i\left(\tfrac{m\pi}{L}\right)^4 + \bar{N}_o^i\bar{h}_y^i\left(\tfrac{m\pi}{L}\right)^2\right)\delta_{mj} \\ \\ + K_1^i\bar{I}_y^i\left(\tfrac{m\pi}{L}\right)^4\bar{q}_{mj}^{(c)} + \bar{Q}_{ymj}^{(c)} + K_3\bar{h}_z^i\left(\tfrac{m\pi}{L}\right)\bar{\eta}_{xymj}^{(c)} \end{bmatrix} \begin{Bmatrix} -\sin\delta_{mj}\eta_i \\ \\ \cos\delta_{mj}\eta_i \end{Bmatrix}\right]$$

$$\begin{Bmatrix} \bar{\Psi}_{m5j+e_m} \\ \\ \bar{\Psi}_{m5j+f_m} \end{Bmatrix} = e^{\gamma_{mj}\eta_i}\left[\begin{bmatrix} \bar{I}_y^i\left(\tfrac{m\pi}{L}\right)^4 - \bar{N}_o^i\left(\tfrac{m\pi}{L}\right)^2 + \left(\bar{d}^i\bar{I}_y^i\left(\tfrac{m\pi}{L}\right)^4 + \bar{N}_o^i\bar{h}_y^i\left(\tfrac{m\pi}{L}\right)^2\right)\gamma_{mj} \\ \\ + K_1^i\bar{I}_y^i\left(\tfrac{m\pi}{L}\right)^4 q_{mj}^{(c)} - Q_{ymj}^{(c)} + K_3\bar{h}_z^i\left(\tfrac{m\pi}{L}\right)^4\eta_{xymj}^{(c)} \end{bmatrix} \begin{Bmatrix} \cos\delta_{mj}\eta_i \\ \\ \sin\delta_{mj}\eta_i \end{Bmatrix}\right.$$

$$\left. + \begin{bmatrix} \left(\bar{d}_1^i\bar{I}_y^i\left(\tfrac{m\pi}{L}\right)^4 + \bar{N}_o^i\bar{h}_y^i\left(\tfrac{m\pi}{L}\right)^2\right)\delta_{mj} \\ \\ + K_1^i\bar{I}_y^i\left(\tfrac{m\pi}{L}\right)^4\bar{q}_{mj}^{(c)} + \bar{Q}_{ymj}^{(c)} + K_3\bar{h}_z^i\left(\tfrac{m\pi}{L}\right)^4\bar{\eta}_{xymj}^{(c)} \end{bmatrix} \begin{Bmatrix} -\sin\delta_{mj}\eta_i \\ \\ \cos\delta_{mj}\eta_i \end{Bmatrix}\right]$$

$$\Phi_{m6j} = [\eta^{(r)}_{ymj} + \bar{h}^{i}_{y}(\tfrac{m\pi}{L})\eta^{(r)}_{xymj}]\, e^{-\alpha_{mj}\bar{\eta}_i} \quad ; \tag{3-40a}$$

$$\Phi_{m6j+R_{1m}} = [\eta^{(r)}_{ymj} - \bar{h}^{i}_{y}(\tfrac{m\pi}{L})\eta^{(r)}_{xymj}]\, e^{\alpha_{mj}\bar{\eta}_i} \quad ;$$

$$\begin{Bmatrix}\Phi_{m6j+a_m}\\ \Phi_{m6j+b_m}\end{Bmatrix} = -\eta^{(im)}_{ymj}\begin{Bmatrix}\cos\beta_{mj}\bar{\eta}_i\\ \sin\beta_{mj}\bar{\eta}_i\end{Bmatrix} + \bar{h}^{i}_{y}(\tfrac{m\pi}{L})\eta^{(im)}_{xymj}\begin{Bmatrix}\sin\beta_{mj}\bar{\eta}_i\\ -\cos\beta_{mj}\bar{\eta}_i\end{Bmatrix} \tag{3-40b}$$

$$\begin{Bmatrix}\Phi_{m6j+c_m}\\ \Phi_{m6j+d_m}\end{Bmatrix} = e^{-\gamma_{mj}\bar{\eta}_i}\Bigg[[\eta^{(c)}_{ymj} + \bar{h}^{i}_{y}(\tfrac{m\pi}{L})\eta^{(c)}_{xymj}]\begin{Bmatrix}\cos\delta_{mj}\bar{\eta}_i\\ \sin\delta_{mj}\bar{\eta}_i\end{Bmatrix} \tag{3-40c}$$

$$+ [\bar{\eta}^{(c)}_{ymj} + \bar{h}^{i}_{y}(\tfrac{m\pi}{L})\bar{\eta}^{(c)}_{xymj}]\begin{Bmatrix}\sin\delta_{mj}\bar{\eta}_i\\ -\cos\delta_{mj}\bar{\eta}_i\end{Bmatrix}\Bigg]$$

$$\begin{Bmatrix}\Phi_{m6j+e_m}\\ \Phi_{m6j+f_m}\end{Bmatrix} = e^{\gamma_{mj}\bar{\eta}_i}\Bigg[[\eta^{(c)}_{ymj} - \bar{h}^{i}_{y}(\tfrac{m\pi}{L})\eta^{(c)}_{xymj}]\begin{Bmatrix}\cos\delta_{mj}\bar{\eta}_i\\ \sin\delta_{mj}\bar{\eta}_i\end{Bmatrix}$$

$$- [\bar{\eta}^{(c)}_{ymj} - \bar{h}^{i}_{y}(\tfrac{m\pi}{L})\bar{\eta}^{(c)}_{xymj}]\begin{Bmatrix}\sin\delta_{mj}\bar{\eta}_i\\ -\cos\delta_{mj}\bar{\eta}_i\end{Bmatrix}\Bigg]$$

$$\bar{\Psi}_{m6j} = [\kappa^{i}_{2}\bar{\bar{I}}^{i}_{z}(\tfrac{m\pi}{L})^4 - \eta^{(r)}_{ymj} - (\bar{\bar{I}}^{i}_{z}(\tfrac{m\pi}{L})^4 - \bar{\bar{N}}^{i}_{o}(\tfrac{m\pi}{L})^2)\, q^{(r)}_{mj} - \bar{h}^{i}_{y}(\tfrac{m\pi}{L})\eta^{(r)}_{xymj} \tag{3-41a}$$

$$- (\bar{d}^{i}_{2}\bar{\bar{I}}^{i}_{z}(\tfrac{m\pi}{L})^4 - \bar{\bar{N}}^{i}_{o}\bar{h}^{i}_{z}(\tfrac{m\pi}{L})^2)\alpha_{mj}\,]\, e^{-\alpha_{mj}\eta_i} \quad ;$$

$$\bar{\Psi}_{m6j+R_{1m}} = [\kappa^{i}_{2}\bar{\bar{I}}^{i}_{z}(\tfrac{m\pi}{L})^4 - \eta^{(r)}_{ymj} + (\bar{\bar{I}}^{i}_{z}(\tfrac{m\pi}{L})^4 - \bar{\bar{N}}^{i}_{o}(\tfrac{m\pi}{L})^2)\, q^{(r)}_{mj} + \bar{h}^{i}_{y}(\tfrac{m\pi}{L})\eta^{(r)}_{xymj}$$

$$+ (\bar{d}^{i}_{2}\bar{\bar{I}}^{i}_{z}(\tfrac{m\pi}{L})^4 - \bar{\bar{N}}^{i}_{o}\bar{h}^{i}_{z}(\tfrac{m\pi}{L})^2)\alpha_{mj}\,]\, e^{\alpha_{mj}\eta_i} \quad ;$$

$$\begin{Bmatrix} \bar{\Psi}_{m6j+a_m} \\ \\ \bar{\Psi}_{m6j+b_m} \end{Bmatrix} = \left[K_2^i \bar{I}_z^i (\tfrac{m\pi}{L})^2 + \eta_{ymj}^{(im)} \right] \begin{Bmatrix} \cos\beta_{mj}\eta_i \\ \\ \sin\beta_{mj}\eta_i \end{Bmatrix} \tag{3-41b}$$

$$+\left[(\bar{I}_z^i (\tfrac{m\pi}{L})^4 - \bar{\bar{N}}_o^i (\tfrac{m\pi}{L})^2) q_{mj}^{(im)} + \bar{h}_y^i (\tfrac{m\pi}{L}) \eta_{xymj}^{(im)} + (\bar{d}_z^i \bar{I}_z^i (\tfrac{m\pi}{L})^4 - \bar{\bar{N}}_o^i \bar{h}_z^i (\tfrac{m\pi}{L})^2) \beta_{mj} \right] \begin{Bmatrix} -\sin\beta_{mj}\eta_i \\ \\ \cos\beta_{mj}\eta_i \end{Bmatrix}$$

$$\begin{Bmatrix} \bar{\Psi}_{m6j+c_m} \\ \\ \bar{\Psi}_{m6j+d_m} \end{Bmatrix} = e^{-\gamma_{mj}\eta_i} \left[-\begin{bmatrix} (\bar{I}_z^i (\tfrac{m\pi}{L})^4 - \bar{\bar{N}}_o^i (\tfrac{m\pi}{L})^2) q_{mj}^{(c)} - k_2^i \bar{I}_z^i (\tfrac{m\pi}{L})^4 + \eta_{ymj}^{(c)} \\ \\ +(\bar{d}_z^i \bar{I}_z^i (\tfrac{m\pi}{L})^4 - \bar{\bar{N}}_o^i \bar{h}_z^i (\tfrac{m\pi}{L})^2) \gamma_{mj} + \bar{h}_y^i (\tfrac{m\pi}{L}) \eta_{xymj}^{(c)} \end{bmatrix} \begin{Bmatrix} \cos\delta_{mj}\eta_i \\ \\ \sin\delta_{mj}\eta_i \end{Bmatrix} \right. \tag{3-41c}$$

$$\left. \begin{bmatrix} (\bar{I}_z^i (\tfrac{m\pi}{L})^4 - \bar{\bar{N}}_o^i (\tfrac{m\pi}{L})^2) \bar{q}_{mj}^{(c)} + (\bar{d}_z^i \bar{I}_z^i (\tfrac{m\pi}{L})^4 \\ \\ -\bar{\bar{N}}_o^i \bar{h}_z^i (\tfrac{m\pi}{L})^2) \delta_{mj} + \bar{\eta}_{ymj}^{(c)} + \bar{h}_y^i (\tfrac{m\pi}{L}) \bar{\eta}_{xymj}^{(c)} \end{bmatrix} \begin{Bmatrix} -\sin\delta_{mj}\eta_i \\ \\ \cos\delta_{mj}\eta_i \end{Bmatrix} \right]$$

$$\begin{Bmatrix} \bar{\Psi}_{m6j+e_m} \\ \\ \bar{\Psi}_{m6j+f_m} \end{Bmatrix} = e^{\gamma_{mj}\eta_i} \left[\begin{bmatrix} (\bar{I}_z^i (\tfrac{m\pi}{L})^4 - \bar{\bar{N}}_o^i (\tfrac{m\pi}{L})^2) q_{mj}^{(c)} + k_2^i \bar{I}_z^i (\tfrac{m\pi}{L}) - \eta_{ymj}^{(c)} \\ \\ +(\bar{d}_z^i \bar{I}_z^i (\tfrac{m\pi}{L})^4 - \bar{\bar{N}}_o^i (\tfrac{m\pi}{L})^2) \gamma_{mj} + \bar{h}_y^i (\tfrac{m\pi}{L}) \bar{\eta}_{xymj}^{(c)} \end{bmatrix} \begin{Bmatrix} \cos\delta_{mj}\eta_i \\ \\ \sin\delta_{mj}\eta_i \end{Bmatrix} \right.$$

$$\left. \begin{bmatrix} (\bar{I}_z^i (\tfrac{m\pi}{L})^4 - \bar{\bar{N}}_o^i (\tfrac{m\pi}{L})^2) \bar{q}_{mj}^{(c)} + (\bar{d}_z^i \bar{I}_z^i (\tfrac{m\pi}{L})^4 \\ \\ -\bar{\bar{N}}_o^i \bar{h}_z^i (\tfrac{m\pi}{L})^2) \delta_{mj} - \bar{\eta}_{ymj}^{(c)} + \bar{h}_y^i (\tfrac{m\pi}{L}) \bar{\eta}_{xymj}^{(c)} \end{bmatrix} \begin{Bmatrix} -\sin\delta_{mj}\eta_i \\ \\ \cos\delta_{mj}\eta_i \end{Bmatrix} \right]$$

$$\Phi_{m7j} = \left[M_{ymj}^{(r)} - K_3(\bar{z}_o^i - \bar{h}_z^i)\eta_{ymj}^{(r)} - (\bar{y}_o^i - \bar{h}_y^i) Q_{ymj}^{(r)} + K_3 \bar{\omega}_N^i (\tfrac{m\pi}{L}) \eta_{xymj}^{(r)} \right] e^{-\alpha_{mj}\bar{\eta}_i} \; ; \tag{3-42a}$$

$$\Phi_{m7j+R_m} = \left[M_{ymj}^{(r)} - K_3(\bar{z}_o^i - \bar{h}_z^i)\eta_{ymj}^{(r)} + (\bar{y}_o^i - \bar{h}_y^i) Q_{ymj}^{(r)} - K_3 \bar{\omega}_N^i (\tfrac{m\pi}{L}) \eta_{xymj}^{(r)} \right] e^{\alpha_{mj}\bar{\eta}_i} ;$$

$$\begin{Bmatrix} \Phi_{m7j+a_m} \\ \\ \Phi_{m7j+b_m} \end{Bmatrix} = \left[M_{ymj}^{(im)} + K_3(\bar{z}_o^i - \bar{h}_z^i)\eta_{ymj}^{(im)} \right] \begin{Bmatrix} \cos\beta_{mj}\bar{\eta}_i \\ \\ \sin\beta_{mj}\bar{\eta}_i \end{Bmatrix} + \begin{bmatrix} (\bar{y}_o^i - \bar{h}_y^i) Q_{ymj}^{(im)} \\ \\ -\bar{\omega}^i K_3 (\tfrac{m\pi}{L}) \eta_{xymj}^{(im)} \end{bmatrix} \begin{Bmatrix} -\sin\beta_{mj}\bar{\eta}_i \\ \\ \cos\beta_{mj}\bar{\eta}_i \end{Bmatrix} \tag{3-42b}$$

$$\begin{Bmatrix} \Phi_{mTj+c_m} \\ \Phi_{mTj+d_m} \end{Bmatrix} = e^{-\gamma_{mj}\bar{\eta}_i} \left[\begin{bmatrix} \mathcal{M}^{(c)}_{ymj} - \kappa_3(\bar{z}^i_0 - \bar{h}^i_z)\mathcal{N}^{(c)}_{ymj} \\ -(\bar{y}^i_0 - \bar{h}^i_y) Q^{(c)}_{ymj} + \kappa_3 \bar{\omega}^i_N (\frac{m\pi}{L}) \mathcal{N}^{(c)}_{xymj} \end{bmatrix} \begin{Bmatrix} \cos \delta_{mj}\bar{\eta}_i \\ \sin \delta_{mj}\bar{\eta}_i \end{Bmatrix} \right. \tag{3-42c}$$

$$\left. + \begin{bmatrix} \bar{\mathcal{M}}^{(c)}_{ymj} - \kappa_3(\bar{z}^i_0 - \bar{h}^i_z)\bar{\mathcal{N}}^{(c)}_{ymj} \\ +(\bar{y}^i_0 - \bar{h}^i_y) \bar{Q}^{(c)}_{ymj} + \kappa_3 \bar{\omega}^i_N (\frac{m\pi}{L}) \bar{\mathcal{N}}^{(c)}_{xymj} \end{bmatrix} \begin{Bmatrix} \sin \delta_{mj}\bar{\eta}_i \\ -\cos \delta_{mj}\bar{\eta}_i \end{Bmatrix} \right]$$

$$\begin{Bmatrix} \Phi_{mTj+e_m} \\ \Phi_{mTj+f_m} \end{Bmatrix} = e^{\gamma_{mj}\bar{\eta}_i} \left[\begin{bmatrix} \mathcal{M}^{(c)}_{ymj} - \kappa_3(\bar{z}^i_0 - \bar{h}^i_z)\mathcal{N}^{(c)}_{ymj} \\ +(\bar{y}^i_0 - \bar{h}^i_y) Q^{(c)}_{ymj} - \kappa_3 \bar{\omega}^i_N (\frac{m\pi}{L}) \mathcal{N}^{(c)}_{xymj} \end{bmatrix} \begin{Bmatrix} \cos \delta_{mj}\bar{\eta}_i \\ \sin \delta_{mj}\bar{\eta}_i \end{Bmatrix} \right.$$

$$\left. + \begin{bmatrix} \bar{\mathcal{M}}^{(c)}_{ymj} - \kappa_3(\bar{z}^i_0 - \bar{h}^i_z)\bar{\mathcal{N}}^{(c)}_{ymj} \\ -(\bar{y}^i_0 - \bar{h}^i_y) \bar{Q}^{(c)}_{ymj} - \kappa_3 \bar{\omega}^i_N (\frac{m\pi}{L}) \bar{\mathcal{N}}^{(c)}_{xymj} \end{bmatrix} \begin{Bmatrix} -\sin \delta_{mj}\bar{\eta}_i \\ \cos \delta_{mj}\bar{\eta}_i \end{Bmatrix} \right]$$

$$\bar{\Psi}_{mTj} = -\Big[\bar{N}^i_0 \bar{y}^i_0 (\tfrac{m\pi}{L})^2 + \mathcal{M}^{(r)}_{ymj} - \kappa_3(\bar{z}^i_0 - \bar{h}^i_z)\mathcal{N}^{(r)}_{ymj} + \bar{N}^i_0 \bar{z}^i_0 (\tfrac{m\pi}{L})^2 q^{(r)}_{mj} \tag{3-43a}$$

$$- (\bar{c}^i_1 (\tfrac{m\pi}{L})^4 + \bar{c}^i_0 (\tfrac{m\pi}{L})^2)\alpha_{mj} - (\bar{y}^i_0 - \bar{h}^i_y) Q^{(r)}_{ymj} + \kappa_3 \bar{\omega}^i_N (\tfrac{m\pi}{L}) \mathcal{N}^{(r)}_{xymj} \Big] e^{-\alpha_{mj}\eta_i}$$

$$\bar{\Psi}_{mTj+k_m} = -\Big[\bar{N}^i_0 \bar{y}^i_0 (\tfrac{m\pi}{L})^2 + \mathcal{M}^{(r)}_{ymj} - \kappa_3(\bar{z}^i_0 - \bar{h}^i_z)\mathcal{N}^{(r)}_{ymj} - \bar{N}^i_0 \bar{z}^i_0 (\tfrac{m\pi}{L})^2 q^{(r)}_{mj}$$

$$+ (\bar{c}^i_1 (\tfrac{m\pi}{L})^4 + \bar{c}^i_0 (\tfrac{m\pi}{L})^2)\alpha_{mj} + (\bar{y}^i_0 - \bar{h}^i_y) Q^{(r)}_{ymj} - \kappa_3 \bar{\omega}^i_N (\tfrac{m\pi}{L}) \mathcal{N}^{(r)}_{xymj} \Big] e^{\alpha_{mj}\eta_i}$$

$$\begin{Bmatrix} \bar{\Psi}_{mTj+a_m} \\ \bar{\Psi}_{mTj+b_m} \end{Bmatrix} = -\Big[\bar{N}^i_0 \bar{y}^i_0 (\tfrac{m\pi}{L})^2 + \mathcal{M}^{(im)}_{ymj} + \kappa_3(\bar{z}^i_0 - \bar{h}^i_z)\mathcal{N}^{(im)}_{ymj} \Big] \begin{Bmatrix} \cos \beta_{mj}\eta_i \\ \sin \beta_{mj}\eta_i \end{Bmatrix} \tag{3-43b}$$

$$+ \begin{bmatrix} \bar{N}^i_0 \bar{z}^i_0 (\frac{m\pi}{L})^2 q^{(im)}_{mj} - (\bar{c}^i_1 (\frac{m\pi}{L})^4 + \bar{c}^i_0 (\frac{m\pi}{L})^2)\beta_{mj} \\ -(\bar{y}^i_0 - \bar{h}^i_y) Q^{(im)}_{ymj} + \kappa_3 \bar{\omega}^i_N (\frac{m\pi}{L}) \mathcal{N}^{(im)}_{xymj} \end{bmatrix} \begin{Bmatrix} -\sin \beta_{mj}\eta_i \\ \cos \beta_{mj}\eta_i \end{Bmatrix}$$

$$\left\{\begin{array}{c}\bar{\Psi}_{m7j+c_m}\\ \\ \bar{\Psi}_{m7j+d_m}\end{array}\right\} = e^{-\gamma_{mj}\eta_i}\left[\left[\begin{array}{c}(\bar{c}_1^i(\frac{m\pi}{L})^4+\bar{c}_0^i(\frac{m\pi}{L})^2)\gamma_{mj} - \bar{N}_0^i\bar{y}_0^i(\frac{m\pi}{L})^2 - \bar{N}_0^i\bar{z}_0^i(\frac{m\pi}{L})^2 q_{mj}^{(c)}\\ \\ -\mathcal{M}_{ymj}^{(c)} + k_3(\bar{z}_0^i-\bar{h}_z^i)\eta_{ymj}^{(c)} + (\bar{y}^i-\bar{h}_y^i)Q_{ymj}^{(c)} - k_3\bar{\omega}_N^i(\frac{m\pi}{L})\eta_{xymj}^{(c)}\end{array}\right]\left\{\begin{array}{c}\cos\delta_{mj}\eta_i\\ \\ \sin\delta_{mj}\eta_i\end{array}\right\}\right.$$

$$\left. + \left[\begin{array}{c}-(\bar{c}_1^i(\frac{m\pi}{L})^4+\bar{c}_0^i(\frac{m\pi}{L})^2)\delta_{mj} + \bar{N}_0^i\bar{z}_0^i(\frac{m\pi}{L})^2\bar{q}_{mj}^{(c)} + \bar{\mathcal{M}}_{ymj}^{(c)}\\ \\ -k_3(\bar{z}_0^i-\bar{h}_z^i)\bar{\eta}_{ymj}^{(c)} + (\bar{y}_0^i-\bar{h}_y^i)\bar{Q}_{ymj}^{(c)} + k_3\bar{\omega}_N^i(\frac{m\pi}{L})\bar{\eta}_{xymj}^{(c)}\end{array}\right]\left\{\begin{array}{c}-\sin\delta_{mj}\eta_i\\ \\ \cos\delta_{mj}\eta_i\end{array}\right\}\right] \tag{3-43c}$$

$$\left\{\begin{array}{c}\bar{\Psi}_{m7j+e_m}\\ \\ \bar{\Psi}_{m7j+f_m}\end{array}\right\} = e^{\gamma_{mj}\eta_i}\left[\left[\begin{array}{c}-(\bar{c}_1^i(\frac{m\pi}{L})^4+\bar{c}_0^i(\frac{m\pi}{L})^2)\gamma_{mj} - \bar{N}_0^i\bar{y}_0^i(\frac{m\pi}{L})^2 + \bar{N}_0^i\bar{z}_0^i(\frac{m\pi}{L})^2 q_{mj}^{(c)}\\ \\ -\mathcal{M}_{ymj}^{(c)} + k_3(\bar{z}_0^i-\bar{h}_z^i)\eta_{ymj}^{(c)} + (\bar{y}_0^i-\bar{h}_y^i)Q_{ymj}^{(c)} - k_3\bar{\omega}_N^i(\frac{m\pi}{L})\eta_{xymj}^{(c)}\end{array}\right]\left\{\begin{array}{c}\cos\delta_{mj}\eta_i\\ \\ \sin\delta_{mj}\eta_i\end{array}\right\}\right.$$

$$\left. \left[\begin{array}{c}-(\bar{c}_1^i(\frac{m\pi}{L})^4+\bar{c}_0^i(\frac{m\pi}{L})^2)\delta_{mj} + \bar{N}_0^i\bar{z}_0^i(\frac{m\pi}{L})^2\bar{q}_{mj}^{(c)} - \bar{\mathcal{M}}_{ymj}^{(c)}\\ \\ + k_3(\bar{z}_0^i-\bar{h}_z^i)\bar{\eta}_{ymj}^{(c)} + (\bar{y}_0^i-\bar{h}_y^i)\bar{Q}_{ymj}^{(c)} + k_3\bar{\omega}_N^i(\frac{m\pi}{L})\bar{\eta}_{xymj}^{(c)}\end{array}\right]\left\{\begin{array}{c}-\sin\delta_{mj}\eta_i\\ \\ \cos\delta_{mj}\eta_i\end{array}\right\}\right]$$

$$\Phi_{m8j} = -\eta_{xymj}^{(r)}\, e^{-\alpha_{mj}\bar{\eta}_i} \quad ; \qquad \Phi_{m8j+R_m} = \eta_{xymj}^{(r)}\, e^{\alpha_{mj}\bar{\eta}_i} \tag{3-44a}$$

$$\Phi_{m8j+a_m} = -\eta_{xymj}^{(im)}\sin\beta_{mj}\bar{\eta}_i \quad ; \qquad \Phi_{m8j+b_m} = \eta_{xymj}^{(im)}\cos\beta_{mj}\bar{\eta}_i \tag{3-44b}$$

$$\left\{\begin{array}{c}\Phi_{m8j+c_m}\\ \\ \Phi_{m8j+d_m}\end{array}\right\} = e^{-\gamma_{mj}\bar{\eta}_i}\left[-\eta_{xymj}^{(c)}\left\{\begin{array}{c}\cos\delta_{mj}\bar{\eta}_i\\ \\ \sin\delta_{mj}\bar{\eta}_i\end{array}\right\} + \bar{\eta}_{xymj}^{(c)}\left\{\begin{array}{c}-\sin\delta_{mj}\bar{\eta}_i\\ \\ \cos\delta_{mj}\bar{\eta}_i\end{array}\right\}\right] \tag{3-44c}$$

$$\left\{\begin{array}{c}\Phi_{m8j+e_m}\\ \\ \Phi_{m8j+f_m}\end{array}\right\} = e^{\gamma_{mj}\bar{\eta}_i}\left[\eta_{xymj}^{(c)}\left\{\begin{array}{c}\cos\delta_{mj}\bar{\eta}_i\\ \\ \sin\delta_{mj}\bar{\eta}_i\end{array}\right\} + \bar{\eta}_{xymj}^{(c)}\left\{\begin{array}{c}-\sin\delta_{mj}\bar{\eta}_i\\ \\ \cos\delta_{mj}\bar{\eta}_i\end{array}\right\}\right]$$

$$\bar{\Psi}_{m8j} = \left[\bar{A}_2^i(\frac{m\pi}{L})^2 p_{mj}^{(r)} + \bar{A}_2^i(\frac{m\pi}{L})^3(\bar{h}_z^i - \bar{h}_y^i q_{mj}^{(r)} - \bar{d}_3^i\alpha_{mj}) - \eta_{xymj}^{(r)}\right] e^{-\alpha_{mj}\eta_i} \quad ; \tag{3-45a}$$

$$\bar{\Psi}_{m8j+R_{1m}} = \left[\bar{A}_2^i \left(\frac{m\pi}{L}\right)^2 P_{mj}^{(r)} + \bar{A}_2^i \left(\frac{m\pi}{L}\right)^3 \left(\bar{h}_z^i - \bar{h}_y^i q_{mj}^{(r)} - \bar{d}_3^i \alpha_{mj} \right) + n_{xymj}^{(r)} \right] e^{\alpha_{mj}\eta_i};$$

$$\begin{Bmatrix} \bar{\Psi}_{m8j+a_m} \\ \bar{\Psi}_{m8j+b_m} \end{Bmatrix} = \bar{A}_2^i \left(\frac{m\pi}{L}\right)^2 \left[P_{mj}^{(im)} + \left(\frac{m\pi}{L}\right) \bar{h}_z^i \right] \begin{Bmatrix} \cos\beta_{mj}\eta_i \\ \sin\beta_{mj}\eta_i \end{Bmatrix} \tag{3-45b}$$

$$+ \left[-n_{xymj}^{(im)} + \bar{A}_2^i \left(\frac{m\pi}{L}\right)^3 \left(\bar{h}_y^i q_{mj}^{(im)} + \bar{d}_3^i \beta_{mj} \right) \right] \begin{Bmatrix} -\sin\beta_{mj}\eta_i \\ \cos\beta_{mj}\eta_i \end{Bmatrix} ;$$

$$\begin{Bmatrix} \bar{\Psi}_{m8j+c_m} \\ \bar{\Psi}_{m8j+d_m} \end{Bmatrix} = e^{-\gamma_{mj}\eta_i} \left[\begin{bmatrix} \bar{A}_2^i \left(\frac{m\pi}{L}\right)^2 \left(P_{mj}^{(c)} + \left(\frac{m\pi}{L}\right) \bar{h}_z^i - \left(\frac{m\pi}{L}\right) \bar{h}_y^i q_{mj}^{(c)} \right. \\ \left. - \left(\frac{m\pi}{L}\right) \bar{d}_3^i \gamma_{mj} \right) + n_{xymj}^{(c)} \end{bmatrix} \begin{Bmatrix} \cos\delta_{mj}\eta_i \\ \sin\delta_{mj}\eta_i \end{Bmatrix} \right. \tag{3-45c}$$

$$\left. + \begin{bmatrix} \bar{A}_2^i \left(\frac{m\pi}{L}\right)^2 \left(\bar{P}_{mj}^{(c)} - \left(\frac{m\pi}{L}\right) \bar{h}_y^i \bar{q}_{mj}^{(c)} \right. \\ \left. - \left(\frac{m\pi}{L}\right) \bar{d}_3^i \delta_{mj} \right) + \bar{n}_{xymj}^{(c)} \end{bmatrix} \begin{Bmatrix} \sin\delta_{mj}\eta_i \\ -\cos\delta_{mj}\eta_i \end{Bmatrix} \right] ;$$

$$\begin{Bmatrix} \bar{\Psi}_{m8j+e_m} \\ \bar{\Psi}_{m8j+f_m} \end{Bmatrix} = e^{\gamma_{mj}\eta_i} \left[\begin{bmatrix} \bar{A}_2^i \left(\frac{m\pi}{L}\right)^2 \left(P_{mj}^{(c)} + \left(\frac{m\pi}{L}\right) \bar{h}_z^i + \left(\frac{m\pi}{L}\right) \bar{h}_y^i q_{mj}^{(c)} \right. \\ \left. - \left(\frac{m\pi}{L}\right) \bar{d}_3^i \delta_{mj} \right) - n_{xymj}^{(c)} \end{bmatrix} \begin{Bmatrix} \cos\delta_{mj}\eta_i \\ \sin\delta_{mj}\eta_i \end{Bmatrix} \right.$$

$$\left. \begin{bmatrix} \bar{A}_2^i \left(\frac{m\pi}{L}\right)^2 \left(\bar{P}_{mj}^{(c)} + \left(\frac{m\pi}{L}\right) \bar{h}_y^i \bar{q}_{mj}^{(c)} \right. \\ \left. + \left(\frac{m\pi}{L}\right) \bar{d}_3^i \delta_{mj} \right) - \bar{n}_{xymj}^{(c)} \end{bmatrix} \begin{Bmatrix} -\sin\delta_{mj}\eta_i \\ \cos\delta_{mj}\eta_i \end{Bmatrix} \right] ;$$

In the previous expressions (Eqs.(3-30) through (3-45)), the subscript "j" ranges from 1 through R_{1m} for part (a), from 1 through R_{2m} for part (b), and from 1 through R_{3m} for part (c) of the equations.

Eqs.(3-29) represent a set of 8N homogeneous algebraic equations with the A^i_j's as independent variables. If the applied axial load is the critical load, this set of equations will have nontrivial solutions when the determinant of the coefficient matrix, which is function of σ, is zero; i.e.,

$$\det.\left[\Omega_m(\sigma)\right] = 0, \text{ for each } m, \tag{3-46}$$

where

$$\left[\Omega_i(\sigma)\right] = \begin{bmatrix} \left[\bar{\Psi}_{m\kappa j}(\eta_1)\right] & \left[\ 0\ \right] & \cdots & \cdots & \cdots & \cdots & \left[\phi_{m\kappa j}(\bar{\eta}_1)\right] \\ \left[\phi_{m\kappa j}(\bar{\eta}_2)\right] & \left[\bar{\Psi}_{m\kappa j}(\eta_2)\right] & \left[\ 0\ \right] & & & & \vdots \\ \vdots & & & & & & \vdots \\ \cdots & \left[\ 0\ \right] & \left[\phi_{m\kappa j}(\bar{\eta}_i)\right] & \left[\bar{\Psi}_{m\kappa j}(\eta_i)\right] & \left[\ 0\ \right] & \cdots & \cdots \\ \vdots & & & & & & \vdots \\ \left[\ 0\ \right] & \cdots & \cdots & \cdots & \left[\ 0\ \right] & \left[\phi_{m\kappa j}(\bar{\eta}_N)\right] & \left[\bar{\Psi}_{m\kappa j}(\eta_N)\right] \end{bmatrix}$$

Thus one needs to find the lowest eigenvalue, σ_m, which satisfies Eq.(3-46), and the critical value, σ_{cr}, is

$$\sigma_{cr} = \min.\left\{\sigma_m\right\}.$$

CHAPTER IV

PRE-BUCKLING DEFORMATION EFFECTS

To rigorously study the pre-buckling deformation effects in longitudinally stiffened cylindrical shells, it is necessary to properly account for the boundary conditions and to use the nonlinear strain-displacement relations. The buckling increment will explicitly coupled with the pre-buckling deformation which is not the case when a membrane pre-buckling state is assumed. The pre-buckling deformation must therefore be solved for first. In this analysis the equations governing the buckling increment are linearized, and then solved by means of the Galerkin method.

The Galerkin method may be briefly summarized as follows: In order to solve a differential equation

$$L(u) = 0 \qquad \text{in a domain } R \tag{4-1}$$

subject to some boundary conditions, a series of functions that satisfy the boundary conditions exactly but with undetermined parameters is assumed: for example

$$\bar{u}_{MN} = \sum_{m=1}^{M} \sum_{n=1}^{N} A_{mn} u_{mn} \ . \tag{4-2}$$

The assumed expression is then substituted into the differential Eq.(4-1). Eq.(4-1) is satisfied if Eq.(4-2) is the exact solution of the problem; otherwise, $L(\bar{u}_{MN})$ is a function of x and y, and therefore can be taken as a measure of the error associated with the approximation $\bar{u}_{MN}$. Multiply $L(\bar{u}_{MN})$ by u_{mn}, integrate the resulting expression over the domain, and then equate it to zero. If there are MN coefficients, MN algebraic equations are obtained which are to be solved simultaneously for A_{mn}'s.

Pre-buckling Deformation

In the case of no external loads in the longitudinal and circumferential directions ($q_x = q_y = 0$), Eqs.(A-23) and (A-24) can be written, by letting u_o, v_o and w_o represent pre-buckling deformation components, as

$$\frac{1}{Eh}\nabla^4 F = \left(\frac{\partial^2 w_o}{\partial x \partial y}\right)^2 - \frac{\partial^2 w_o}{\partial x^2}\frac{\partial^2 w_o}{\partial y^2} - \frac{1}{a}\frac{\partial^2 w_o}{\partial x^2} , \tag{4-3}$$

and

$$D\nabla^4 w_o = q + \frac{1}{a}\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2}\frac{\partial^2 w_o}{\partial x^2} - 2\frac{\partial^2 F}{\partial x \partial y}\frac{\partial^2 w_o}{\partial x \partial y} + \frac{\partial^2 F}{\partial x^2}\frac{\partial^2 w_o}{\partial y^2} , \tag{4-4}$$

where q is the interacting normal loading between the skin and stringers. It is very difficult to solve these equations exactly, because the stringers eliminate the simplifying axisymmetric response. Therefore, an iterative method is used.

For the first iteration, the constant stress resultant N_o, and linearized differential equations governing the pre-buckling deformation are considered, so that

$$N_x = -N_o = -ph \quad , \tag{4-5}$$

and Eq.(4-4) becomes

$$D\nabla^4 w_o = -\frac{1}{a}N_y + q \quad . \tag{4-6}$$

By introducing the linear stress-displacement relations

$$N_x = \frac{Eh}{1-\nu^2}\left[\frac{\partial u_o}{\partial x} + \nu\left(\frac{\partial v_o}{\partial y} - \frac{w_o}{a}\right)\right] , \tag{4-7}$$

$$N_y = \frac{Eh}{1-\nu^2}\left[\frac{\partial v_o}{\partial y} - \frac{w_o}{a} + \nu\frac{\partial u_o}{\partial y}\right] , \tag{4-8}$$

the displacement component u_o can, from Eq.(4-7), be expressed in the equation

$$\frac{\partial u_o}{\partial x} = -\frac{N_o(1-\nu^2)}{Eh} + \nu\frac{w_o}{a} - \nu\frac{\partial v_o}{\partial y} . \tag{4-9}$$

Substitutions of Eq.(4-9) into Eq.(4-8) results in

$$N_y = -\nu N_o + Eh\left(\frac{\partial v_o}{\partial y} - \frac{w_o}{a}\right) . \tag{4-10}$$

Finally, substitution of Eq.(4-10) into Eq.(4-6) yields

$$D\nabla^4 w_o = -\frac{\nu}{a}N_o + q + \frac{Eh}{a}\left(\frac{\partial v_o}{\partial y} - \frac{w_o}{a}\right) , \tag{4-11}$$

where the displacement component, v_o, satisfies Eq.(A-32)

$$a\nabla^4 v_o = (2+\nu)\frac{\partial^3 w_o}{\partial x^2 \partial y} + \frac{\partial^3 w_o}{\partial y^3} . \tag{A-32}$$

By introducing the nondimensional quantities as given in Eq.(2-36), Eq.(4-11) and Eq.(A-32) can be written as

$$\frac{1}{\rho^2}\bar{\nabla}^4\bar{w}_o + \bar{w}_o - \frac{\partial \bar{v}_o}{\partial \eta} = -\nu(2\sigma^2) + q \,, \qquad (4\text{-}12)$$

and

$$\bar{\nabla}^4 \bar{v}_o = (2+\nu)\frac{\partial^3 \bar{w}_o}{\partial \xi^2 \partial \eta} + \frac{\partial^3 \bar{w}_o}{\partial \eta^3} \,, \qquad (4\text{-}13)$$

where

$$\bar{\nabla}^2 = \frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2} .$$

Then Eqs.(4-12) and (4-13) govern the pre-buckling deformation.

For the case of equally spaced symmetric stringers, the radial deformation is symmetric about the stringer locations and middle section of the cylinder, and the stringers deform only in the x-z plane. The governing differential equation for the stringer is then

$$\bar{I}_y \frac{d^4 \bar{w}_{os}}{d \xi^{*4}} + \bar{N}_o \frac{d^2 \bar{w}_{os}}{d \xi^{*2}} = \bar{V}_{ys}(\xi^*) . \qquad (4\text{-}14)$$

Let the interacting normal force between stringer and skin, $\bar{V}_{ys}(\xi^*)$, be expanded in Fourier series as

$$\bar{V}_{ys}(\xi^*) = \sum_{m=1,3,\cdots}^{\infty} \rho^2 q_m \sin \frac{m\pi \xi^*}{L} \qquad (4\text{-}15)$$

then the deflection of the stringer, $\bar{w}_{os}$, is

$$\bar{w}_{os} = \sum_{m=1,3,\cdots}^{\infty} \frac{\rho^2 q_m}{\left[\bar{I}_y\left(\frac{m\pi}{L}\right)^4 - \bar{N}_o\left(\frac{m\pi}{L}\right)^2\right]} \sin \frac{m\pi \xi^*}{L} \qquad (4\text{-}16)$$

for the simply supported case.

For convenience, the coordinates are set as shown in Figure 4. Then shell deformations are assumed to be in the form

$$\bar{w}_c = \sum_{m=1,3}^{\infty} \sum_{n=0}^{\infty} W_{mn} \sin \frac{m\pi\xi^*}{L} \cos \frac{2n\pi\eta^*}{d} \; ; \tag{4-17}$$

$$\bar{v}_c = \sum_{m=1,3}^{\infty} \sum_{n=0}^{\infty} V_{mn} \sin \frac{m\pi\xi^*}{L} \sin \frac{2n\pi\eta^*}{d} \; , \tag{4-18}$$

which satisfy the boundary conditions, and "d" is the distance between two adjacent stringers divided by the shell radius, "a". Substitution of Eqs.(4-17) and (4-18) into Eq.(4-13) yields

$$V_{mn} = \frac{(2+\nu)\left(\frac{m\pi}{L}\right)^2\left(\frac{2n\pi}{d}\right) + \left(\frac{2n\pi}{d}\right)^3}{\left[\left(\frac{m\pi}{L}\right)^2 + \left(\frac{2n\pi}{d}\right)^2\right]^2} W_{mn} \; , \tag{4-19}$$

and

$$V_{mo} = 0 \; . \tag{4-20}$$

Expand the loading terms into double Fourier series as

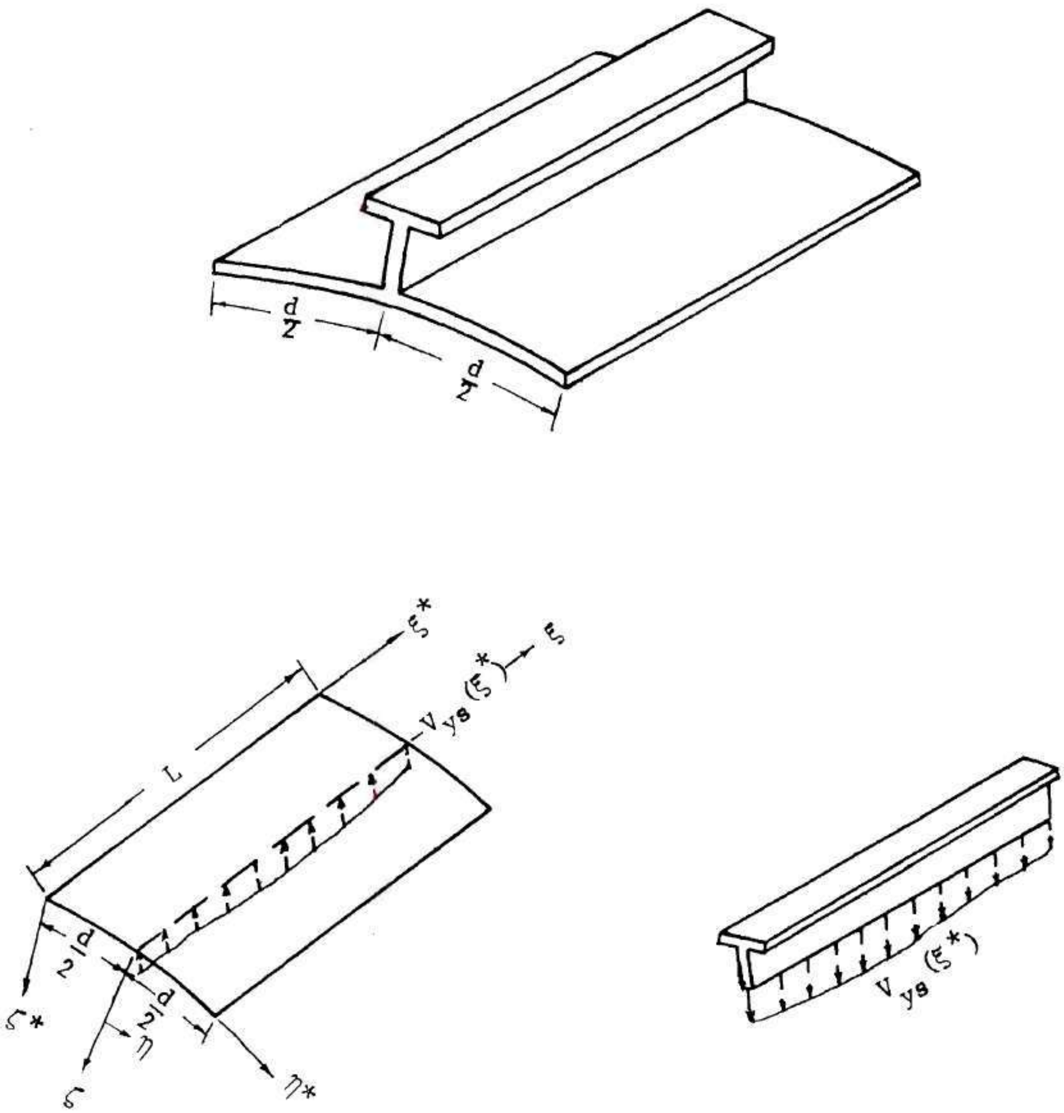
$$-2\sigma^2\nu = \sum_{m=1,3\ldots}^{\infty} a_{mo} \sin \frac{m\pi\xi^*}{L} \; , \tag{4-21}$$

and

$$\bar{q}(\xi^*,\eta^*) = -\frac{1}{\rho^2} \bar{V}_{ys}(\xi^*)\, \delta(\eta^* - \tfrac{d}{2}) \tag{4-22}$$

$$= \sum_{m=1,3}^{\infty} \sum_{n=0}^{\infty} p_{mn} \sin \frac{m\pi\xi^*}{L} \cos \frac{2n\pi\eta^*}{d} \; ,$$

where



Shell

Stringer

Figure 4. Typical Section for Calculating
Pre-buckling Deformation

$$a_{mo} = -\frac{8\sigma^2\nu}{m\pi L} \quad ; \quad p_{mn} = -\frac{2q_m}{d}\cos n\pi \quad ; \quad p_{mo} = -\frac{q_m}{d} .$$

By substituting Eqs.(4-17) through (4-22) into Eq.(4-12) and then solving for W_{mn}, one may express the radial displacement in terms of q_m and σ as

$$\bar{w}_o = \sum_{m=1,3}^{\infty} \left[-\frac{\left(\frac{q_m}{d} + \frac{8\sigma^2\nu}{m\pi}\right)}{D_{mo}} - \sum_{n=1}^{\infty} \frac{2q_m \cos n\pi}{D_{mn}} \cos\frac{2n\pi\eta^*}{d} \right] \sin\frac{m\pi\xi^*}{L} , \tag{4-23}$$

where

$$D_{mo} = \frac{1}{\rho^2}\left(\frac{m\pi}{L}\right)^4 + 1 ;$$

$$D_{mn} = \frac{1}{\rho^2}\left[\left(\frac{m\pi}{L}\right)^2 + \left(\frac{2n\pi}{d}\right)^2\right]^2 + 1 + \left(\frac{2n\pi}{d}\right) \frac{(2+\nu)\left(\frac{m\pi}{L}\right)^2\left(\frac{2n\pi}{d}\right) + \left(\frac{2n\pi}{d}\right)^3}{\left[\left(\frac{m\pi}{L}\right)^2 + \left(\frac{2n\pi}{d}\right)^2\right]^2} .$$

By matching the displacements of the shell, Eq.(4-23), and the stringer, Eq.(4-16), along the intersecting line, one obtains the expression for q_m

$$q_m = \frac{-\frac{8\sigma^2\nu}{m\pi L D_{mo}}}{\frac{\rho^2}{\left[\bar{I}_y\left(\frac{m\pi}{L}\right)^4 - \bar{N}_o\left(\frac{m\pi}{L}\right)^2\right]} + \frac{1}{d D_{mo}} + \sum_{n=1}^{\infty} \frac{2}{D_{mn}}(\cos n\pi)^2} . \tag{4-24}$$

After transforming the coordinates, pre-buckling deformation can be expressed as

$$\bar{w}_o = \sum_{m=1,3}^{\infty} \sum_{n=0}^{\infty} A_{mn} \cos\frac{2n\pi\left(\eta - \frac{d}{2}\right)}{d} \sin\frac{m\pi\xi}{L} , \tag{4-25}$$

where

$$A_{mo} = \frac{\frac{\gamma_m}{d} + \frac{8\sigma^2\nu}{m\pi L}}{D_{mo}} \quad ; \qquad A_{mn} = -\frac{\frac{2\gamma_m}{d}\cos n\pi}{D_{mn}}$$

It is noted that $\bar{v}_o$ is zero and $\bar{w}_o$ has zero slope in the circumferential direction along the intersection lines between the shell and stringers; each stringer is thus bent in the x-z plane only and the obtained pre-buckling deformation, $\bar{w}_o$, given in Eq.(4-25) satisfies the continuity conditions and compatibility conditions.

The radial displacement for the first iteration can be expressed as

$$w_o^{(1)} = \sum_{m=o}^{\infty}\sum_{n=o}^{\infty} G_{mn} \sin\frac{m\pi x^*}{\ell} \cos\frac{2n\pi y^*}{ad} \tag{4-26}$$

where $G_{mn} = aA_{mn}$ are constants for given axial load. Neglecting the non-linear terms in w_o, Eq.(4-3) becomes

$$\frac{1}{Eh}\nabla^4 F^{(1)} = -\frac{1}{a}\frac{\partial^2 w_o^{(1)}}{\partial x^{*2}} \tag{4-27}$$

Substituting Eq.(4-26) into Eq.(4-27) and then solving for the stress function, $F^{(2)}$, one obtains

$$F^{(2)} = -\frac{1}{2}N_o y^{*2} - \frac{\nu}{2}N_o x^{*2} + \sum_{m=o}^{\infty}\sum_{n=o}^{\infty} F_{mn} \sin\frac{m\pi x^{*2}}{\ell}\cos\frac{2n\pi y^*}{ad} \tag{4-28}$$

where

$$F_{mn} = \frac{Eh\left(\frac{m\pi}{L}\right)^2 G_{mn}}{a\left[\left(\frac{m\pi}{L}\right)^2 + \left(\frac{2n\pi}{d}\right)^2\right]^2}$$

Note that the boundary conditions on F are considered to be $N_x = -N_o$ and $N_y = -\nu N_o$ at $x = 0$ and $x = \ell$. Hence

$$\frac{N_x}{Eh} = \frac{1}{Eh}\frac{\partial^2 F^{(2)}}{\partial y^{*2}} = -\frac{N_o}{Eh} - \sum_{m=1,3\cdots}^{\infty}\sum_{n=1}^{\infty} N_{mn} \sin\frac{m\pi x^*}{\ell}\cos\frac{2n\pi y^*}{d},$$

where

$$N_{mn} = \frac{\left(\frac{m\pi}{L}\right)^4 A_{mn}}{\left[\left(\frac{m\pi}{L}\right)^2 + \left(\frac{2n\pi}{d}\right)^2\right]^2}.$$

Numerical calculations based on geometry and material used in the numerical examples presented later in Chapter V show that

$$\frac{|N_x - N_o|}{|N_o|} < 10^{-2}$$

and the convergence is very rapid; hence the solution of the first iteration is good enough.

For further iterations, one may substitute Eqs.(4-26) and (4-28) into the right hand side of Eq.(4-4), which becomes

$$D\nabla^4 w_o^{(2)} = q + \frac{1}{a}\frac{\partial^2 F^{(2)}}{\partial x^{*2}} + \frac{\partial^2 F^{(2)}}{\partial y^{*2}}\frac{\partial^2 w_o^{(1)}}{\partial x^{*2}} - 2\frac{\partial^2 F^{(2)}}{\partial x^*\partial y^*}\frac{\partial^2 w_o^{(1)}}{\partial x^*\partial y^*} + \frac{\partial^2 F^{(2)}}{\partial x^{*2}}\frac{\partial^2 w_o^{(1)}}{\partial y^{*2}} \qquad (4\text{-}29)$$

and then solve for $w_o^{(2)}$ to get a more exact solution.

Determination of Critical Load

At the critical value of axial load, let

$$\bar{w} = \bar{w}_o + \bar{w}' \quad , \quad \bar{N}_x = \bar{N}_{ox} + \bar{N}'_x \quad , \tag{4-30}$$

$$\bar{N}_y = \bar{N}_{oy} + \bar{N}'_y \quad , \quad \bar{N}_{xy} = \bar{N}_{oxy} + \bar{N}'_{xy} \quad ,$$

where $\bar{w}_o$, $\bar{N}_{ox}$, $\bar{N}_{oy}$ and $\bar{N}_{oxy}$ are the radial displacement and stress resultants in the pre-buckling state, respectively, and $\bar{w}'$, $\bar{N}'_x$, $\bar{N}'_y$ and $\bar{N}'_{xy}$ are perturbation quantities. The linearized differential equation governing the buckling increment, Eq.(A-35), can be written in the dimensionless form as

$$\frac{1}{\rho^2}\bar{\nabla}^8\bar{w}' + \frac{\partial^4\bar{w}'}{\partial\xi^4} - \frac{1}{1-\nu^2}\bar{\nabla}^4\Big(\bar{N}_{ox}\frac{\partial^2\bar{w}'}{\partial\xi^2} + 2\bar{N}_{oxy}\frac{\partial^2\bar{w}'}{\partial\xi\partial\eta} \tag{4-31}$$

$$+ \bar{N}_{oy}\frac{\partial^2\bar{w}'}{\partial\eta^2} + \bar{N}'_x\frac{\partial^2\bar{w}_o}{\partial\xi^2} + 2\bar{N}'_{xy}\frac{\partial^2\bar{w}_o}{\partial\xi\partial\eta} + \bar{N}'_y\frac{\partial^2\bar{w}_o}{\partial\eta^2}\Big)$$

$$= \frac{\partial^2}{\partial\xi^2}\Big(2\frac{\partial^2\bar{w}_o}{\partial\xi\partial\eta}\frac{\partial^2\bar{w}'}{\partial\xi\partial\eta} - \frac{\partial^2\bar{w}_o}{\partial\xi^2}\frac{\partial^2\bar{w}'}{\partial\eta^2} - \frac{\partial^2\bar{w}_o}{\partial\eta^2}\frac{\partial^2\bar{w}'}{\partial\xi^2}\Big) .$$

The products of the stress resultants, $\bar{N}_{oy}$ and $\bar{N}_{oxy}$, with the derivatives of the radial displacement increment are neglected, since either these stress resultants are very small as compared with $\bar{N}_{ox}$ or the derivative of $\bar{w}'$ in the circumferential direction is very small. Therefore Eq.(4-31) in conjunction with Eq.(4-5) becomes

$$\frac{1}{\rho^2}\bar{\nabla}^8\bar{w}' + 2\sigma^2\bar{\nabla}^4\Big(\frac{\partial^2\bar{w}'}{\partial\xi^2}\Big) + \frac{\partial^4\bar{w}'}{\partial\xi^4} = \frac{\partial^2}{\partial\xi^2}\Big(2\frac{\partial^2\bar{w}_o}{\partial\xi\partial\eta}\frac{\partial^2\bar{w}'}{\partial\xi\partial\eta} \tag{4-32}$$

$$- \frac{\partial^2\bar{w}_o}{\partial\xi^2}\frac{\partial^2\bar{w}'}{\partial\eta^2} - \frac{\partial^2\bar{w}_o}{\partial\eta^2}\frac{\partial^2\bar{w}'}{\partial\xi^2}\Big) + \frac{1}{1-\nu^2}\bar{\nabla}^4\Big(\bar{N}'_x\frac{\partial^2\bar{w}_o}{\partial\xi^2} + 2\bar{N}'_{xy}\frac{\partial^2\bar{w}_o}{\partial\xi\partial\eta} + \bar{N}'_y\frac{\partial^2\bar{w}_o}{\partial\eta^2}\Big) ,$$

Eq.(4-32) is a linear partial differential equation with variable coefficient, which contains the pre-buckling displacement component, w_o, as given in Eq.(4-25), and governs the buckling increment w'. The stress resultant increments, N'_x , N'_y and N'_{xy}, can be expressed in terms of w' by means of Eqs.(2-40), (2-41) and (2-42) in conjunction with Eq.(3-6) and (3-7) when all these equations are expressed in terms of buckling increments.

To get a series of functions which satisfy the boundary conditions exactly, one may consider the expression

$$\bar{w}'(\xi,\eta) = \sum_{m=1}^{\infty} B_m \bar{w}'_m(\xi,\eta) = \sum_{m=1}^{\infty} B_m \sin\frac{m\pi\xi}{L} W'_m(\eta) \qquad (4\text{-}33)$$

This form of radial displacement increment will exactly satisfy the classical boundary conditions of simply supported case at $\xi=0$ and $\xi=L$. It will be very difficult to assume the form of the function $W'_m(\eta)$ which satisfies all the conditions given in Eqs.(2-48) through (2-55). It is therefore necessary to generate a set of function usable in the Galerkin method. Ideally, these functions should give as small an error term as possible when substituted into Eq.(4-32), but require as little work as possible for generation of the functions. These conflicting requirements can be satisfied by developing the set of eigenfunctions corresponding to the following differential equation:

$$\frac{1}{\rho^2}\bar{\nabla}^8 \bar{w}'_m + 2\epsilon_m^2 \bar{\nabla}^4 \frac{\partial^2 \bar{w}'_m}{\partial \xi^2} + \frac{\partial^4 \bar{w}'_m}{\partial \xi^4} = 0 . \qquad (4\text{-}34)$$

Eq.(4-34) has the same form as Eq.(3-1) except that the eigenvalue term $2\epsilon_m^2$ replaces the loading term $2\sigma^2$, and the procedure for generating these eigenvalues is similar to that used in the linear analysis presented in Chapter III. There is one difference, however. In Chapter III, the eigenvalue σ appears in the compatibility equations, Eqs.(2-52) through (2-55), while the pre-buckling axial force N_o is a specified quantity in the compatibility equations associated with Eq.(3-4). In other words, the eigenvalue ϵ_m corresponding to Eq.(4-35) occurs only in the differential equation, while the eigenvalue σ corresponding to critical load occurs in both differential equation and the boundary conditions. Therefore the values of ϵ_m determined from Eq.(4-34) will in general be different from the values of σ_m obtained in Chapter III.

The solution to Eq. (4-34) is assumed in the form

$$\bar{w}_m'(\xi,\eta) = \sin\frac{m\pi\xi}{L}\, W_m'(\eta) \,. \tag{4-35}$$

Substitution into Eq.(4-34) gives the ordinary differential equation

$$\frac{1}{\rho^2}\left[\frac{d^2}{d\eta^2} - \left(\frac{m\pi}{L}\right)^2\right]^4 W_m'(\eta) - 2\epsilon_m^2\left(\frac{m\pi}{L}\right)^2\left[\frac{d^2}{d\eta^2} - \left(\frac{m\pi}{L}\right)^2\right]^2 W_m'(\eta) + \left(\frac{m\pi}{L}\right)^4 W_m'(\eta) = 0, \tag{4-36}$$

As in the linear analysis presented in Chapter III, $W_m'(\eta)$ can be solved in terms of eight unknown constants

$$\begin{aligned} W_m'^{i}(\eta) = {} & \sum_{j=1}^{R_{1m}} \left(\bar{A}^{i}_{mj}\, e^{-\alpha_{mj}\eta} + \bar{A}^{i}_{mj+R_{1m}}\, e^{\alpha_{mj}\eta}\right) + \sum_{j=1}^{R_{3m}} \left(\bar{A}^{i}_{mj+a_m}\cos\beta_{mj}\eta\right. \\ & \left. + \bar{A}^{i}_{mj+b_m}\sin\beta_{mj}\eta\right) + \sum_{j=1}^{R_{3m}} \left(\bar{A}^{i}_{mj+c_m}\, e^{-\gamma_{mj}\eta}\cos\delta_{mj}\eta\right. \\ & + \bar{A}^{i}_{mj+d_m}\, e^{-\gamma_{mj}\eta}\sin\delta_{mj}\eta + \bar{A}^{i}_{mj+e_m}\, e^{\gamma_{mj}\eta}\cos\delta_{mj}\eta \\ & \left. + \bar{A}^{i}_{mj+f_m}\, e^{\gamma_{mj}\eta}\sin\delta_{mj}\eta\right). \end{aligned} \tag{4-37}$$

where $\pm\alpha_{mj}$, $\pm i\beta_{mj}$ and $\pm\gamma_{mj}\pm i\delta_{mj}$ are the roots of the characteristic equation

$$\frac{1}{\rho^2}\left[\lambda_m^2-\left(\frac{m\pi}{L}\right)^2\right]^4-2\epsilon_m^2\left(\frac{m\pi}{L}\right)^2\left[\lambda_m^2-\left(\frac{m\pi}{L}\right)^2\right]^2+\left(\frac{m\pi}{L}\right)^4=0\,. \tag{4-38}$$

Eq.(4-38) has the same form as Eq.(3-11) with ϵ_m^2 replacing σ^2, and $2R_{1m}$, $2R_{2m}$ and $4R_{3m}$ are respectively the number of the real, pure imaginary, and complex roots of the characteristic equation, Eq.(4-38), and a_m, b_m, c_m, d_m, e_m and f_m are defined in Eq.(3-18).

Similar to the linear case, the expressions for the displacement increments $\bar{v}'$ and $\bar{u}'$, can be obtained as

$$\begin{aligned}\bar{v}^i_m(\xi,\eta)=\sin\frac{m\pi\xi}{L}\Big\{&\sum_{j=1}^{R_{1m}}\left[-\bar{A}^i_{mj}\,q^{(r)}_{mj}\,e^{-\alpha_{mj}\eta}+\bar{A}^i_{mj+R_{1m}}\,q^{(r)}_{mj}\,e^{\alpha_{mj}\eta}\right]\\
&+\sum_{j=1}^{R_{2m}}\left[-\bar{A}^i_{mj+a_m}\,q^{(im)}_{mj}\sin\beta_{mj}\eta+\bar{A}^i_{mj+b_m}\,q^{(im)}_{mj}\cos\beta_{mj}\eta\right]\\
&+\sum_{j=1}^{R_{3m}}\Big[\bar{A}^i_{mj+c_m}\,e^{-\gamma_{mj}\eta}\left(-q^{(c)}_{mj}\cos\delta_{mj}\eta-\bar{q}^{(c)}_{mj}\sin\delta_{mj}\eta\right)\\
&+\bar{A}^i_{mj+d_m}\,e^{-\gamma_{mj}\eta}\left(-q^{(c)}_{mj}\sin\delta_{mj}\eta+\bar{q}^{(c)}_{mj}\cos\delta_{mj}\eta\right)\\
&+\bar{A}^i_{mj+e_m}\,e^{\gamma_{mj}\eta}\left(q^{(c)}_{mj}\cos\delta_{mj}\eta-\bar{q}^{(c)}_{mj}\sin\delta_{mj}\eta\right)\\
&+\bar{A}^i_{mj+f_m}\,e^{\gamma_{mj}\eta}\left(q^{(c)}_{mj}\sin\delta_{mj}\eta+\bar{q}^{(c)}_{mj}\cos\delta_{mj}\eta\right)\Big]\Big\}\,,\end{aligned} \tag{4-39}$$

$$\bar{u}^i_m(\xi,\eta)=\cos\frac{m\pi\xi}{L}\Big\{\sum_{j=1}^{R_{1m}}\left[\bar{A}^i_{mj}\,p^{(r)}_{mj}\,e^{-\alpha_{mj}\eta}+\bar{A}^i_{mj+R_{1m}}\,p^{(r)}_{mj}\,e^{\alpha_{mj}\eta}\right] \tag{4-40}$$

$$+ \sum_{j=1}^{R_{2m}} [\bar{A}^i_{mj+a_m} p^{(im)}_{mj} \cos\beta_{mj}\eta + \bar{A}^i_{mj+b_m} p^{(im)}_{mj} \sin\beta_{mj}\eta]$$

$$+ \sum_{j=1}^{R_{3m}} [\bar{A}^i_{mj+c_m} e^{-\gamma_{mj}\eta} (p^{(c)}_{mj} \cos\delta_{mj}\eta + \bar{p}^{(c)}_{mj} \sin\delta_{mj}\eta)$$

$$+ \bar{A}^i_{mj+d_m} e^{-\gamma_{mj}\eta} (p^{(c)}_{mj} \sin\delta_{mj}\eta - \bar{p}^{(c)}_{mj} \cos\delta_{mj}\eta)$$

$$+ \bar{A}^i_{mj+e_m} e^{\gamma_{mj}\eta} (p^{(c)}_{mj} \cos\delta_{mj}\eta - \bar{p}^{(c)}_{mj} \sin\delta_{mj}\eta)$$

$$+ \bar{A}^i_{mj+f_m} e^{\gamma_{mj}\eta} (p^{(c)}_{mj} \sin\delta_{mj}\eta + \bar{p}^{(c)}_{mj} \cos\delta_{mj}\eta)]\} ,$$

where $p^{(r)}_{mj}$, $p^{(im)}_{mj}$, $p^{(c)}_{mj}$ and $\bar{p}^{(c)}_{mj}$, and $q^{(r)}_{mj}$, $q^{(im)}_{mj}$, $q^{(c)}_{mj}$ and $\bar{q}^{(c)}_{mj}$ are defined as in Eqs.(3-19) and (3-20), respectively, with the term σ^2 replaced by the term ϵ_m^2.

Substitution of Eqs.(4-37), (4-39) and (4-40) into Eqs.(2-40) through (2-47) and then into Eqs.(2-48) through (2-55) results in the simultaneous linear homogeneous equations,

$$\begin{bmatrix} [\bar{\Psi}_{mkj}(\eta_1)] & [\ 0\ ] & \cdots & \cdots & [\Phi_{mkj}(\bar{\eta}_1)] \\ [\Phi_{mkj}(\bar{\eta}_2)] & [\bar{\Psi}_{mkj}(\eta_2)] & [\ 0\ ] & \cdots & [\ 0\ ] \\ \vdots & & & & \vdots \\ \cdots & [\ 0\ ] & [\Phi_{mkj}(\bar{\eta}_i)] & [\bar{\Psi}_{mkj}(\eta_i)] & [\ 0\ ] \cdots \\ \vdots & & & & \vdots \\ [\ 0\ ] & \cdots & [\ 0\ ] & [\Phi_{mkj}(\bar{\eta}_N)] & [\bar{\Psi}_{mkj}(\eta_N)] \end{bmatrix} \begin{bmatrix} [\bar{A}^1_{mj}] \\ [\bar{A}^2_{mj}] \\ \vdots \\ [\bar{A}^i_{mj}] \\ \vdots \\ [\bar{A}^N_{mj}] \end{bmatrix} = [\,0\,] \quad (4\text{-}41)$$

where ϕ_{mkj} and Ψ_{mkj} are as given in Eqs.(3-30) through (3-45). It is noted that N_o is function of σ, not of ϵ_m, in the equations of compatibility. Thus Eq.(4-41) is not function of ϵ_m explicitly and the solution of Eq.(4-41) is different from that of Eq.(3-46).

In order to get the nontrivial solutions, the determinant of the coefficient matrix is set to zero and hence a high order nonlinear equation for ϵ_m is obtained. Solving this equation, one may obtain a set of roots, $\epsilon_{m1} < \epsilon_{m2} \cdots < \epsilon_{mn} \cdots$, for each m, and the corresponding basic eigenvectors are

$$\left[a_{mn}\right] = \begin{bmatrix} \left[a^1_{mn}\right] \\ \cdot \\ \cdot \\ \left[a^i_{mn}\right] \\ \cdot \\ \cdot \\ \left[a^N_{mn}\right] \end{bmatrix} \quad \text{where} \quad \left[a^i_{mn}\right] = \begin{bmatrix} a^i_{mn1} \\ a^i_{mn2} \\ \cdot \\ \cdot \\ \cdot \\ a^i_{mn8} \end{bmatrix} \quad \text{and} \quad \sum_{i=1}^{N}\sum_{j=1}^{8} (a^i_{mnj})^2 = 1. \tag{4-42}$$

Thus, the desired series of functions which satisfy the boundary conditions exactly can finally be constructed as

$$\bar{w}^i_{(\xi,\eta)} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} B_{mn} W'^i_{mn}(\eta) \sin\frac{m\pi\xi}{L} \quad \text{for } i = 1,2 \ldots N \tag{4-43}$$

where

$$W'^i_{mn}(\eta) = \sum_{j=1}^{R_{1mn}} \left(a^i_{mnj} e^{-\alpha_{mnj}\eta} + a^i_{mnj+R_{1mn}} e^{\alpha_{mnj}\eta}\right) + \sum_{j=1}^{R_{2mn}} \left(a^i_{mnj+a_{mn}} \cos\beta_{mnj}\eta + a^i_{mnj+b_{mn}} \sin\beta_{mnj}\eta\right)$$
$$+ \sum_{j=1}^{R_{3m}} \left(a^i_{mnj+c_{mn}} e^{-\gamma_{mnj}\eta} \cos\delta_{mnj}\eta + a^i_{mnj+d_{mn}} e^{-\gamma_{mnj}\eta} \sin\delta_{mnj}\eta + a^i_{mnj+e_{mn}} e^{\gamma_{mnj}\eta} \cos\delta_{mnj}\eta + a^i_{mnj+f_{mn}} e^{\gamma_{mnj}\eta} \sin\delta_{mnj}\eta\right),$$

in which $\pm\alpha_{mnj}$, $\pm i\beta_{mnj}$ and $\pm\gamma_{mnj} \pm i\,\delta_{mnj}$ are the roots of the characteristic equation

$$\frac{1}{\rho^2}\left[\lambda^2 - \left(\frac{m\pi}{L}\right)^2\right]^4 - 2\epsilon_{mn}^2\left[\lambda^2 - \left(\frac{m\pi}{L}\right)^2\right]^2\left(\frac{m\pi}{L}\right)^2 + \left(\frac{m\pi}{L}\right)^4 = 0; \qquad (4\text{-}44)$$

$2R_{1mn}$, $2R_{2mn}$ and $4R_{3mn}$ are the number of the real, pure imaginary, and complex roots of the characteristic equation, respectively, and

$$a_{mn} = 2R_{mn}\ , \quad b_{mn} = a_{mn} + R_{2mn}\ , \quad c_{mn} = b_{mn} + 2R_{mn}\ ,$$

$$d_{mn} = c_{mn} + R_{3mn}\ , \quad e_{mn} = d_{mn} + R_{3mn}\ , \quad f_{mn} = e_{mn} + R_{3mn}\ .$$

It is noted that the radial displacement increment for the i^{th} panel given in Eq.(4-43) satisfies the boundary conditions at both ends, as well as the equations of continuity and equations of compatibility along the stringers exactly, but not necessarily the governing differential equation, Eq.(4-32), in general. Hence, by the Galerkin method, it is required that

$$\sum_{i=1}^{N}\int_{\eta_{i-1}}^{\eta_i}\int_0^L \mathcal{L}(\bar{w}'^{i})\sin\frac{k\pi\xi}{L}\,W_{kl}^{i}(\eta)\,d\xi\,d\eta = 0\ , \qquad (4\text{-}45)$$

for $k = 1,2. \ldots$, and $l = 1,2 \ldots$,

where

$$\mathcal{L}(\bar{w}') = \frac{1}{\rho^2}\bar{\nabla}^8\bar{w}' + 2\sigma^2\bar{\nabla}^4\frac{\partial^2\bar{w}'}{\partial\xi^2} + \frac{\partial^4\bar{w}'}{\partial\xi^4} - 2\frac{\partial^4\bar{w}_0}{\partial\xi^3\partial\eta}\frac{\partial^2\bar{w}'}{\partial\xi\partial\eta} - 4\frac{\partial^3\bar{w}_0}{\partial\xi^2\partial\eta}\frac{\partial^3\bar{w}'}{\partial\xi^2\partial\eta}$$

$$- 2\frac{\partial^2\bar{w}_0}{\partial\xi\partial\eta}\frac{\partial^4\bar{w}'}{\partial\xi^3\partial\eta} + \frac{\partial^4\bar{w}_0}{\partial\xi^4}\frac{\partial^2\bar{w}'}{\partial\eta^2} + 2\frac{\partial^3\bar{w}_0}{\partial\xi^3}\frac{\partial^3\bar{w}'}{\partial\xi\partial\eta^2} + \frac{\partial^2\bar{w}_0}{\partial\xi^2}\frac{\partial^4\bar{w}'}{\partial\xi^2\partial\eta^2} + \frac{\partial^4\bar{w}_0}{\partial\xi^2\partial\eta^2}\frac{\partial^2\bar{w}'}{\partial\xi^2}$$

$$+ 2\frac{\partial^3 \bar{w}_o}{\partial\xi\partial\eta^2}\frac{\partial^3 \bar{w}'}{\partial\xi^3} + \frac{\partial^2 \bar{w}_o}{\partial\eta^2}\frac{\partial^4 \bar{w}'}{\partial\xi^4} + \frac{1}{1-\nu^2}\Big[\bar{\mathcal{L}}\Big(\bar{N}_x', \frac{\partial^2 \bar{w}_o}{\partial\xi^2}\Big)$$

$$+ 2\bar{\mathcal{L}}\Big(\bar{N}_{xy}', \frac{\partial^2 \bar{w}_o}{\partial\xi\partial\eta}\Big) + \bar{\mathcal{L}}\Big(\bar{N}_y', \frac{\partial^2 \bar{w}_o}{\partial\eta^2}\Big)\Big],$$

in which

$$\bar{\mathcal{L}}(f,g) = f\cdot\frac{\partial^4 g}{\partial\xi^4} + 4\frac{\partial f}{\partial\xi}\frac{\partial^3 g}{\partial\xi^3} + 6\frac{\partial^2 f}{\partial\xi^2}\frac{\partial^2 g}{\partial\xi^2} + 4\frac{\partial^3 f}{\partial\xi^3}\frac{\partial g}{\partial\xi} + \frac{\partial^4 f}{\partial\xi^4}\cdot g$$

$$+ f\cdot\frac{\partial^4 g}{\partial\xi^2\partial\eta^2} + 2\frac{\partial f}{\partial\eta}\frac{\partial^3 g}{\partial\xi^2\partial\eta} + \frac{\partial^2 f}{\partial\eta^2}\frac{\partial^2 g}{\partial\xi^2} + 2\frac{\partial^3 f}{\partial\xi\partial\eta^2}\frac{\partial g}{\partial\xi} + 4\frac{\partial^2 f}{\partial\xi\partial\eta}\frac{\partial^2 g}{\partial\xi\partial\eta}$$

$$+ 2\frac{\partial f}{\partial\xi}\frac{\partial^3 g}{\partial\xi\partial\eta^2} + \frac{\partial^4 f}{\partial\xi^2\partial\eta^2}\cdot g + 2\frac{\partial^3 f}{\partial\xi^2\partial\eta}\frac{\partial g}{\partial\eta} + \frac{\partial^2 f}{\partial\xi^2}\frac{\partial^2 g}{\partial\eta^2} + f\cdot\frac{\partial^4 g}{\partial\eta^4}$$

$$+ 4\frac{\partial f}{\partial\eta}\frac{\partial^3 g}{\partial\eta^3} + 6\frac{\partial^2 f}{\partial\eta^2}\frac{\partial^2 g}{\partial\eta^2} + 4\frac{\partial^3 f}{\partial\eta^3}\frac{\partial g}{\partial\eta} + \frac{\partial^4 f}{\partial\eta^4}\cdot g .$$

Recalling that

$$\int_0^L \sin\frac{m\pi\xi}{L}\sin\frac{n\pi\xi}{L}\,d\xi = \frac{L}{2}\delta_{mn}, \qquad (4\text{-}46)$$

$$\int_0^L \sin\frac{m\pi\xi}{L}\sin\frac{n\pi\xi}{L}\sin\frac{k\pi\xi}{L}\,d\xi = ISIN_{mnk}$$

$$= IS1_{mnk} + IS2_{mnk} - IS3_{mnk} + IS4_{mnk},$$

$$\int_0^L \cos\frac{m\pi\xi}{L}\cos\frac{n\pi\xi}{L}\sin\frac{k\pi\xi}{L}\,d\xi = ICOS_{mnk}$$

$$= IS1_{mnk} + IS2_{mnk} + IS3_{mnk} - IS4_{mnk},$$

where

$$IS1_{mnk} = \begin{cases} \dfrac{L}{4} & \text{if } (k+m-n) = 0 \\ 0 & \text{if } (k+m-n) \text{ is even} \\ \dfrac{L}{2(k+m-n)\pi} & \text{if } (k+m-n) \text{ is odd} \end{cases}$$

$$IS2_{mnk} = \begin{cases} \dfrac{L}{4} & \text{if } (k-m+n) = 0 \\ 0 & \text{if } (k-m+n) \text{ is even} \\ \dfrac{L}{2(k-m+n)\pi} & \text{if } (k-m+n) \text{ is odd} \end{cases}$$

$$IS3_{mnk} = \begin{cases} 0 & \text{if } (k+m+n) \text{ is even} \\ \dfrac{L}{2(k+m+n)\pi} & \text{if } (k+m+n) \text{ is odd} \end{cases}$$

$$IS4_{mnk} = \begin{cases} \dfrac{L}{4} & \text{if } (m+n-k) = 0 \\ 0 & \text{if } (m+n-k) \text{ is even} \\ \dfrac{L}{2(m+n-k)\pi} & \text{if } (m+n-k) \text{ is odd} \end{cases}$$

Eq.(4-45) yields the following simultaneous algebraic equations for B_{mn}'s:

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} K_{k\ell mn} B_{mn} = 0 \tag{4-47}$$

for $k = 1, 2 \ . \ . \ .$ and $\ell = 1, 2 \ . \ . \ .$

where

$$K_{k\ell mn} = 2\left(\frac{m\pi}{L}\right)^2(\epsilon_m^2-\sigma^2)\ \frac{L}{2}\ C_{3k\ell mn}\ \delta_{km} \tag{4-48}$$

$$+\sum_{m'=1,3}^{\infty}\sum_{n'=0}^{\infty} A_{m'n'}\Big\{2\ ICOS_{mm'k}\Big[-C_{1k\ell mnm'n'}\ \frac{m\pi}{L}\ \frac{m'\pi}{L}\left(\left(\frac{m\pi}{L}\right)^2+\left(\frac{m\pi}{L}\right)^2\right)$$

$$+\ C_{0k\ell mnm'n'}\left(\frac{m\pi}{L}\right)^3\frac{m'\pi}{L}\ -\ C_{2k\ell mnm'n'}\ \frac{m\pi}{L}\left(\frac{m'\pi}{L}\right)^3\Big]$$

$$+\ ISIN_{mm'k}\Big[4\ C_{1k\ell mnm'n'}\left(\frac{m\pi}{L}\right)^2\left(\frac{m'\pi}{L}\right)^2\ +\ C_{2k\ell mnm'n'}\left(\frac{m'\pi}{L}\right)^2\cdot$$

$$\cdot\left(\left(\frac{m\pi}{L}\right)^2+\left(\frac{m'\pi}{L}\right)^2\right)\ -\ C_{0k\ell mnm'n'}\left(\frac{m\pi}{L}\right)^2\left(\left(\frac{m\pi}{L}\right)^2+\left(\frac{m'\pi}{L}\right)^2\right)\Big]$$

$$+\ \frac{1}{1-\nu^2}\ D_{k\ell mnm'n'}\Big\},$$

in which

$$C_{0k\ell mnm'n'} = \left(\frac{2n'\pi}{d}\right)^2\sum_{i=1}^{N}\int_{\eta_{i-1}}^{\eta_i} W'^{i}_{mn}(\eta)\cdot\cos\frac{2n'\pi(\eta-\frac{d}{2})}{d}\ W'^{i}_{k\ell}(\eta)\ d\eta\ ;$$

$$C_{1k\ell mnm'n'} = \left(\frac{2n'\pi}{d}\right)^2\sum_{i=1}^{N}\int_{\eta_{i-1}}^{\eta_i}\frac{dW'^{i}_{mn}}{d\eta}\ \sin\frac{2n'\pi(\eta-\frac{d}{2})}{d}\ W'^{i}_{k\ell}(\eta)\ d\eta\ ;$$

$$C_{2k\ell mnm'n'} = \sum_{i=1}^{N}\int_{\eta_{i-1}}^{\eta_i}\frac{d^2W'^{i}_{mn}}{d\eta^2}\ \cos\frac{2n'\pi(\eta-\frac{d}{2})}{d}\ W'^{i}_{k\ell}(\eta)\ d\eta\ ;$$

$$C_{3k\ell mn} = \sum_{i=1}^{N}\int_{\eta_{i-1}}^{\eta_i}\left(\frac{d^4W'^{i}_{mn}}{d\eta^4}\ -\ 2\left(\frac{m\pi}{L}\right)^2\frac{d^2W'^{i}_{mn}}{d\eta^2}\ +\ \left(\frac{m\pi}{L}\right)^4W'^{i}_{mn}\right]W'^{i}_{k\ell}(\eta)\ d\eta\ ;$$

$$D_{k\ell mnm'n'} = \left\{ \sum_{i=1}^{N} \int_{\eta_i}^{\eta_i} E^{i}_{mn}(\eta) \cos \frac{2n\pi'(\eta - d/2)}{d} W'^{i}_{k\ell}(\eta)\, d\eta \right\} \left\{ 2\ ICOS_{mm'k} \left(\frac{m\pi}{L}\right)\left(\frac{m'\pi}{L}\right)^3 \cdot \right.$$

$$\left[2\left(\frac{m'\pi}{L}\right)^2 + 2\left(\frac{m\pi}{L}\right)^2 + \left(\frac{2n'\pi}{d}\right)^2 \right] - ISIN_{mm'k} \left(\frac{m'\pi}{L}\right)^2 \left[\left(\frac{m'\pi}{L}\right)^4 + 6\left(\frac{m\pi}{L}\right)^2\left(\frac{m'\pi}{L}\right)^2 \right.$$

$$\left. \left. + \left(\frac{m\pi}{L}\right)^4 + \left(\frac{m'\pi}{L}\right)^2\left(\frac{2n'\pi}{d}\right)^2 + \left(\frac{m\pi}{L}\right)^2\left(\frac{2n'\pi}{d}\right)^2 + \left(\frac{2n'\pi}{d}\right)^4 \right] \right\} +$$

$$\left\{ \sum_{i=1}^{N} \int_{\eta_{i-1}}^{\eta_i} \frac{dE^{i}_{mn}(\eta)}{d\eta} \sin \frac{2n'\pi(\eta - d/2)}{d} W'^{i}_{k\ell}(\eta)\, d\eta \right\} \left\{ 4\left(\frac{m\pi}{L}\right)\left(\frac{m'\pi}{L}\right)^3\left(\frac{2n'\pi}{d}\right) ICOS_{mm'k} \right.$$

$$\left. + 2\ ISIN_{mm'k} \left(\frac{m'\pi}{L}\right)^2\left(\frac{2n'\pi}{d}\right) \left[2\left(\frac{2n'\pi}{d}\right)^2 - \left(\frac{m'\pi}{L}\right)^2 - \left(\frac{m\pi}{L}\right)^2 \right] \right\} +$$

$$\left\{ \sum_{i=1}^{N} \int_{\eta_{i-1}}^{\eta_i} \frac{d^2E^{i}_{mn}(\eta)}{d\eta^2} \cos \frac{2n'\pi(\eta - d/2)}{d} W'_{k\ell}(\eta)\, d\eta \right\} \left\{ ISIN_{mm'k} \left(\frac{m'\pi}{L}\right)^2 \left[\left(\frac{m'\pi}{L}\right)^2 + \left(\frac{m\pi}{L}\right)^2 \right. \right.$$

$$\left. \left. + 6\left(\frac{2n'\pi}{d}\right)^2 \right] - 2\left(\frac{m\pi}{L}\right)\left(\frac{m'\pi}{L}\right)^3 ICOS_{mm'k} \right\} + \left\{ \sum_{i=1}^{N} \int_{\eta_{i-1}}^{\eta_i} \frac{d^3E^{i}_{mn}(\eta)}{d\eta^3} \sin \frac{2n'\pi(\eta - d/2)}{d} W'^{i}_{k\ell}(\eta)\, d\eta \right\}$$

$$\left\{ 4\left(\frac{m'\pi}{L}\right)^2\left(\frac{2n'\pi}{d}\right)^2 ISIN_{mm'k} \right\} - \left\{ \sum_{i=1}^{N} \int_{\eta_{i-1}}^{\eta_i} \frac{d^4E^{i}_{mn}(\eta)}{d\eta^4} \cos \frac{2n'\pi(\eta - d/2)}{d} W'^{i}_{k\ell}(\eta)\, d\eta \right\} \left\{ \left(\frac{m'\pi}{L}\right)^2 ISIN_{mm'k} \right\}$$

$$+ \left\{ \sum_{i=1}^{N} \int_{\eta_{i-1}}^{\eta_i} F^{i}_{mn}(\eta) \cos \frac{2n'\pi(\eta - d/2)}{d} W'^{i}_{k\ell}(\eta)\, d\eta \right\} \left\{ 2\left(\frac{m\pi}{L}\right)\left(\frac{m'\pi}{L}\right)\left(\frac{2n'\pi}{d}\right)^2 ICOS_{mm'k} \left[2\left(\frac{m'\pi}{L}\right)^2 \right. \right.$$

$$\left. + 2\left(\frac{m\pi}{L}\right)^2 + \left(\frac{2n'\pi}{d}\right)^2 \right] - ISIN_{mm'k} \left(\frac{2n'\pi}{d}\right)^2 \left[\left(\frac{m'\pi}{L}\right)^4 + 6\left(\frac{m\pi}{L}\right)^2\left(\frac{m'\pi}{L}\right)^2 + \left(\frac{m\pi}{L}\right)^4 \right.$$

$$\left. \left. + \left(\frac{m'\pi}{L}\right)^2\left(\frac{m\pi}{L}\right)^2 + \left(\frac{m\pi}{L}\right)^2\left(\frac{2n'\pi}{d}\right)^2 + \left(\frac{2n'\pi}{d}\right)^4 \right] \right\} + \left\{ \sum_{i=1}^{N} \int_{\eta_{i-1}}^{\eta_i} \frac{dF^{i}_{mn}(\eta)}{d\eta} \sin \frac{2n'\pi(\eta - d/2)}{d} W'^{i}_{k\ell}(\eta)\, d\eta \right\}$$

$$\left\{ 4\left(\frac{m\pi}{L}\right)\left(\frac{m'\pi}{L}\right)\left(\frac{2n'\pi}{d}\right)^3 ICOS_{mm'k} - 2\ ISIN_{mm'k} \left(\frac{2n'\pi}{d}\right)^3 \left[\left(\frac{m'\pi}{L}\right)^2 + \left(\frac{m\pi}{L}\right)^2 + 2\left(\frac{2n'\pi}{d}\right)^2 \right] \right\}$$

$$+ \left\{ \sum_{i=1}^{N} \int_{\eta_{i-1}}^{\eta_i} \frac{d^2F^{i}_{mn}(\eta)}{d\eta^2} \cos \frac{2n'\pi(\eta - d/2)}{d} W'^{i}_{k\ell}(\eta)\, d\eta \right\} \left\{ ISIN_{mm'k} \left(\frac{2n'\pi}{d}\right)^2 \left[\left(\frac{m'\pi}{L}\right)^2 + \left(\frac{m\pi}{L}\right)^2 + \right. \right.$$

$$\left. \left. 6\left(\frac{2n'\pi}{d}\right)^2 \right] - 2\left(\frac{m\pi}{L}\right)\left(\frac{m'\pi}{L}\right)\left(\frac{2n'\pi}{d}\right)^2 ICOS_{mm'k} \right\} + \left\{ \sum_{i=1}^{N} \int_{\eta_{i-1}}^{\eta_i} \frac{d^3F^{i}_{mn}(\eta)}{d\eta^3} \sin \frac{2n'\pi(\eta - d/2)}{d} W'^{i}_{k\ell}(\eta)\, d\eta \right\}$$

$$\left\{4\left(\frac{2n'\pi}{d}\right)^3 ISIN_{mm'K}\right\} - \left\{\sum_{i=1}^{N}\int_{\eta_i}^{\eta_i}\frac{d^4F^i_{mn}(\eta)}{d\eta}\cos\frac{2n'\pi(\eta-d/2)}{d}W'^{i}_{K\ell}(\eta)d\eta\right\}\left\{\left(\frac{2n'\pi}{d}\right)^2 ISIN_{mm'K}\right\}$$

$$+2\left\{\sum_{i=1}^{N}\int_{\eta_{i-1}}^{\eta_i}G^i_{mn}(\eta)\sin\frac{2n'\pi(\eta-d/2)}{d}W'^{i}_{K\ell}(\eta)\,d\eta\right\}\left\{2\,ISIN_{mm'K}\left(\frac{m\pi}{L}\right)\left(\frac{m'\pi}{L}\right)\left(\frac{2n'\pi}{d}\right)\left[\left(\frac{m'\pi}{L}\right)^3\right.\right.$$

$$+2\left(\frac{m\pi}{L}\right)^2\left(\frac{m'\pi}{L}\right)+\left(\frac{m'\pi}{L}\right)\left(\frac{2n'\pi}{d}\right)^2\Big] - ICOS_{mm'K}\left(\frac{n'\pi}{L}\right)\left(\frac{2n'\pi}{d}\right)\Big[\left(\frac{m'\pi}{L}\right)^4+6\left(\frac{m\pi}{L}\right)^2\left(\frac{m'\pi}{L}\right)^2+\left(\frac{m\pi}{L}\right)^4$$

$$+\left(\frac{2n'\pi}{d}\right)^4+\left(\frac{m\pi}{L}\right)^2\left(\frac{2n'\pi}{d}\right)^2+\left(\frac{m'\pi}{L}\right)^2\left(\frac{2n'\pi}{d}\right)^2\Big]\Big\} + 2\left\{\sum_{i=1}^{N}\int_{\eta_{i-1}}^{\eta_i}\frac{dG^i_{mn}(\eta)}{d\eta}\cos\frac{2n'\pi(\eta-d/2)}{d}W'^{i}_{K\ell}(\eta)d\eta\right\}$$

$$\left\{2\,ICOS_{mm'K}\left(\frac{2n'\pi}{d}\right)^2\left[\left(\frac{m'\pi}{L}\right)^3+\left(\frac{m\pi}{L}\right)^2\left(\frac{m'\pi}{L}\right)+2\left(\frac{m'\pi}{L}\right)^3\right] - 4\left(\frac{m\pi}{L}\right)\left(\frac{m'\pi}{L}\right)^2\left(\frac{2n'\pi}{d}\right)^2 ISIN_{mm'K}\right\}$$

$$+2\left\{\sum_{i=1}^{N}\int_{\eta_{i-1}}^{\eta_i}\frac{d^2G^i_{mn}(\eta)}{d\eta^2}\sin\frac{2n'\pi(\eta-d/2)}{d}W'^{i}_{K\ell}(\eta)d\eta\right\}\left\{ICOS_{mm'K}\left(\frac{2n'\pi}{d}\right)\left[\left(\frac{m'\pi}{L}\right)^3+\left(\frac{m\pi}{L}\right)^2\left(\frac{m'\pi}{L}\right)\right.\right.$$

$$+6\left(\frac{m'\pi}{L}\right)\left(\frac{2n'\pi}{d}\right)^2\Big] - 2\left(\frac{m\pi}{L}\right)\left(\frac{m'\pi}{L}\right)^2\left(\frac{2n'\pi}{d}\right)\Big\} + 8\left\{\sum_{i=1}^{N}\int_{\eta_{i-1}}^{\eta_i}\frac{d^3G^i_{mn}(\eta)}{d\eta^3}\cos\frac{2n'\pi(\eta-d/2)}{d}W'^{i}_{K\ell}(\eta)d\eta\right\}\cdot$$

$$\left\{\left(\frac{m'\pi}{L}\right)\left(\frac{2n'\pi}{d}\right)^2 ICOS_{mm'K}\right\} - 2\left\{\sum_{i=1}^{N}\int_{\eta_{i-1}}^{\eta_i}\frac{d^4G^i_{mn}(\eta)}{d\eta^4}\sin\frac{2n'\pi(\eta-d/2)}{d}W^{i}_{K\ell}(\eta)d\eta\right\}\left\{\left(\frac{m'\pi}{L}\right)\left(\frac{2n'\pi}{d}\right)ICOS_{mm'K}\right\},$$

where

$$E^i_{mn}(\eta) = \sum_{j=1}^{R_{1mn}}\left[a^i_{mnj}\eta^{(r)}_{xmnj}e^{-\alpha_{mnj}\eta} + a^i_{mnj+R_{1mn}}\eta^{(r)}_{xmnj}e^{\alpha_{mnj}\eta}\right] + \sum_{j=1}^{R_{2mn}}\left[a^i_{mnj+a_{mn}}\eta^{(im)}_{xmnj}\cos\beta_{mnj}\eta\right. +$$

$$a^i_{mnj+b_{mn}}\eta^{(im)}_{xmnj}\sin\beta_{mnj}\eta\Big] + \sum_{j=1}^{R_{3mn}}\Big[a^i_{mnj+c_{mn}}e^{-\delta_{mnj}\eta}\left(\eta^{(c)}_{xmnj}\cos\delta_{mnj}\eta + \bar{\eta}^{(c)}_{xmnj}\sin\delta_{mnj}\eta\right) +$$

$$a^i_{mnj+d_{mn}}e^{-\gamma_{mnj}}\left(\eta^{(c)}_{xmnj}\sin\delta_{mnj}\eta - \bar{\eta}^{(c)}_{xmnj}\cos\delta_{mnj}\right) + a^i_{mnj+e_{mn}}e^{\gamma_{mnj}\eta}\left(\eta^{(c)}_{xmnj}\cos\delta_{mnj}\eta\right. -$$

$$\left.\bar{\eta}^{(c)}_{xmnj}\sin\delta_{mnj}\eta\right) + a^i_{mnj+f_{mn}}e^{\gamma_{mnj}\eta}\left(\eta^{(c)}_{xmnj}\sin\delta_{mnj}\eta + \bar{\eta}^{(c)}_{xmnj}\cos\delta_{mnj}\eta\right)\Big] ;$$

$$F^i_{mn}(\eta) = \sum_{j=1}^{R_{1mn}}\left[a^i_{mnj}\eta^{(r)}_{ymnj}e^{-\alpha_{mnj}\eta} + a^i_{mnj+R_{1mn}}\eta^{(r)}_{ymnj}e^{\alpha_{mnj}\eta}\right] + \sum_{j=1}^{R_{2mn}}\left[a^i_{mnj+a_{mn}}\eta^{(im)}_{ymnj}\cos\beta_{mnj}\eta\right. +$$

$$a^{i}_{mnj+b_{mn}} N^{(im)}_{ymnj} \sin \beta_{mnj}\eta \Big] + \sum_{j=1}^{R_{3mn}} \Big[a^{i}_{mnj+c_{mn}} e^{-\gamma_{mnj}\eta} \left(N^{(c)}_{ymnj} \cos \delta_{mnj}\eta + \bar{N}^{(c)}_{ymnj} \sin \delta_{mnj}\eta \right) +$$

$$a^{i}_{mnj+d_{mn}} e^{-\gamma_{mnj}\eta} \left(N^{(c)}_{ymnj} \sin \delta_{mnj}\eta - \bar{N}^{(c)}_{ymnj} \cos \delta_{mnj}\eta \right) + a^{i}_{mnj+e_{mn}} e^{\gamma_{mnj}\eta} \left(N^{(c)}_{ymnj} \cos \delta_{mnj}\eta - \right.$$

$$\left. \bar{N}^{(c)}_{ymnj} \sin \delta_{mnj}\eta \right) + a^{i}_{mnj+f_{mn}} e^{\gamma_{mnj}\eta} \left(N^{(c)}_{ymnj} \sin \delta_{mnj}\eta + \bar{N}^{(c)}_{ymnj} \cos \delta_{mnj}\eta \right) \Big] \ ;$$

$$G^{i}_{mn}(\eta) = \sum_{j=1}^{R_{1mn}} \Big[-a^{i}_{mnj} N^{(r)}_{xymnj} e^{-\alpha_{mnj}\eta} + a^{i}_{mnj+R_{1mn}} N^{(r)}_{xymnj} e^{\alpha_{mnj}\eta} \Big] + \sum_{j=1}^{R_{2mn}} \Big[-a^{i}_{mnj+a_{mn}} N^{(im)}_{xymnj} \sin \beta_{mnj}\eta +$$

$$a^{i}_{mnj+b_{mn}} N^{(im)}_{xymnj} \cos \beta_{mnj}\eta \Big] + \sum_{j=1}^{R_{3mn}} \Big[a^{i}_{mnj+c_{mn}} e^{-\gamma_{mnj}\eta} \left(-N^{(c)}_{xymnj} \cos \delta_{mnj}\eta - \bar{N}^{(c)}_{xymnj} \sin \delta_{mnj}\eta \right) +$$

$$a^{i}_{mnj+d_{mn}} e^{-\gamma_{mnj}\eta} \left(-N^{(c)}_{xymnj} \sin \delta_{mnj}\eta + \bar{N}^{(c)}_{xymnj} \cos \delta_{mnj}\eta \right) + a^{i}_{mnj+e_{mn}} e^{\gamma_{mnj}\eta} \left(N^{(c)}_{xymnj} \cos \delta_{mnj}\eta - \right.$$

$$\left. \bar{N}^{(c)}_{xymnj} \sin \delta_{mnj}\eta \right) + a^{i}_{mnj+f_{mn}} e^{\gamma_{mnj}\eta} \left(N^{(c)}_{xymnj} \sin \delta_{mnj}\eta + \bar{N}^{(c)}_{xymnj} \cos \delta_{mnj}\eta \right) \Big] \ ,$$

in which

$$N^{(r)}_{xmnj} = \nu \left(q^{(r)}_{mnj} \alpha_{mnj} - 1 \right) - \frac{m\pi}{L} p^{(r)}_{mnj} \ ;$$

$$N^{(im)}_{xmnj} = \nu \left(q^{(im)}_{mnj} \beta_{mnj} + 1 \right) + \frac{m\pi}{L} p^{(im)}_{mnj} \ ;$$

$$N^{(c)}_{xmnj} = \nu \left(\gamma_{mnj} q^{(c)}_{mnj} - \delta_{mnj} \bar{q}^{(c)}_{mnj} - 1 \right) - \frac{m\pi}{L} p^{(c)}_{mnj} \ ;$$

$$\bar{N}^{(c)}_{xmnj} = \nu \left(\gamma_{mnj} \bar{q}^{(c)}_{mnj} + \delta_{mnj} q^{(c)}_{mnj} \right) - \frac{m\pi}{L} \bar{p}^{(c)}_{mnj} \ ;$$

$$N^{(r)}_{ymnj} = q^{(r)}_{mnj} \alpha_{mnj} - 1 - \nu \frac{m\pi}{L} p^{(r)}_{mnj} \ ;$$

$$N^{(im)}_{ymnj} = q^{(im)} \beta_{mnj} + 1 + \nu \frac{m\pi}{L} p^{(im)}_{mnj} \ ;$$

$$n^{(c)}_{ymnj} = \gamma_{mnj}\, q^{(c)}_{mnj} - \delta_{mnj}\, \bar{q}^{(c)}_{mnj} - 1 - \nu \frac{m\pi}{L} p^{(c)}_{mnj} \quad ;$$

$$\bar{n}^{(c)}_{ymnj} = \gamma_{mnj}\, \bar{q}^{(c)}_{mnj} + \delta_{mnj}\, q^{(c)}_{mnj} - \nu \frac{m\pi}{L} \bar{p}^{(c)}_{mnj} \quad ;$$

$$n^{(r)}_{xymnj} = \tfrac{1}{2}(1-\nu^2)(p^{(r)}_{mnj}\alpha_{mnj} + \tfrac{m\pi}{L} q^{(r)}_{mnj}) ;$$

$$n^{(im)}_{xymnj} = \tfrac{1}{2}(1-\nu^2)(p^{(im)}_{mnj}\beta_{mnj} + \tfrac{m\pi}{L} q^{(im)}_{mnj}) ;$$

$$n^{(c)}_{xymnj} = \tfrac{1}{2}(1-\nu^2)(p^{(c)}_{mnj}\gamma_{mnj} - \bar{p}^{(c)}_{mnj}\delta_{mnj} + \tfrac{m\pi}{L} q^{(c)}_{mnj}) ;$$

$$\bar{n}^{(c)}_{xymnj} = \tfrac{1}{2}(1-\nu^2)(\bar{p}^{(c)}_{mnj}\gamma_{mnj} + p^{(c)}_{mnj}\delta_{mnj} + \tfrac{m\pi}{L} \bar{q}^{(c)}_{mnj}) ;$$

where $p^{(r)}_{mnj}$, $p^{(im)}_{mnj}$, $p^{(c)}_{mnj}$, $p^{(c)}_{mnj}$, $q^{(r)}_{mnj}$, $q^{(im)}_{mnj}$, $q^{(c)}_{mnj}$ and $q^{(c)}_{mnj}$ as given in Eq.(4-40) with the terms α_{mj} , β_{mj} , γ_{mj} and δ_{mj} replaced by the terms α_{mnj} , β_{mnj} , γ_{mnj} and δ_{mnj} respectively. e.g., $p^{(r)}_{mnj}$ is given by the equation for $p^{(r)}_{mj}$ with the indicated replacements.

If the applied load $(2\sigma^2)$ is the critical load, Eq.(4-47) will have the nontrivial solution and the determinant of the coefficient matrix must be zero; i.e.,

$$\det. \begin{bmatrix} K_{1111} & K_{1112} & \cdots K_{111n} & \cdots K_{1121} & \cdots K_{112n} & \cdots K_{11mn} & \cdots \\ K_{1211} & K_{1212} & \cdots K_{121n} & \cdots K_{1221} & \cdots K_{122n} & \cdots K_{12mn} & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\ K_{1\ell 11} & K_{1\ell 12} & \cdots K_{1\ell 1n} & \cdots K_{1\ell 21} & \cdots K_{1\ell 2n} & \cdots K_{1\ell mn} & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\ K_{k\ell 11} & K_{k\ell 12} & \cdots K_{k\ell 1n} & \cdots K_{k\ell 21} & \cdots K_{k\ell 2n} & \cdots K_{k\ell mn} & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \end{bmatrix} = 0 \qquad (4\text{-}49)$$

The lowest eigenvalue gives the desires critical load.

CHAPTER V

DISCUSSION OF RESULTS

In these numerical calculations, the computer programs were written in both ALGOL and FORTRAN languages for use with the Burroughs' B 5500 and UNIVAC 1108 digital computers. The existing procedures supplied by Burroughs and Univac companies, respectively, are utilized for calculating the determinant value and for solving the simultaneous equations.

Linear Analysis

In the linear analysis, the lowest eigenvalue, σ_m, for each given m is determined first by a trial-and-error method from Eq.(3-48). The lowest σ_m will represent the eigenvalue corresponding to the buckling load, or

$$\sigma_{cr} = min.\{\sigma_m\}.$$

The critical stress carried by skin is

$$\frac{p_{cr}}{E} = 2\,\sigma_{cr}^2 .$$

In order to make a partial check on the computer programming, stringers of very small sizes are considered. The longitudinally stiffened cylindrical shell can then be considered as a moncoque thin cylindrical shell which has a closed form solution for the critical load that may be expressed as

$$p_{cr} = \frac{Eh}{a\sqrt{3(1-\nu^2)}} \quad . \tag{5-1}$$

The results based upon a shell having two, five and twelve stringers yield very good agreement with the classical solution according to Eq.(5-1) with a maximum discrepancy of less than 0.7%. The properties of the stringers are:

$$L = 3 \quad ; \quad \frac{a}{h} = 402 \quad ; \quad \bar{E}^i = 1 \quad ; \quad \nu = 0.3 \quad ;$$

$$\frac{A_s}{ah} = 0.01 \quad ; \quad \frac{I_y}{ah^3} = 0.005 \quad ; \quad \frac{J}{ah^3} = 0.005 \; ,$$

and

$$\bar{C}_1 = \bar{C} = \bar{I}_z = \bar{z}_o = \bar{y}_o = \bar{h}_z = \bar{h}_y = 0 \; .$$

Very limited experimental results are available for cylindrical shells stiffened with stringers placed at large spacings. Tsao and Ching[32] have recently presented some experimental findings for reasonably long ($\frac{\ell}{a}$ = 3) cylindrical shell stiffened by only a few (6, 9, 12, 18) stringers. However, the shell specimens were fixed along both edges. Earlier experimental results for shorter ($\frac{\ell}{a}$ = 1.56) and also fixed stiffened shell are given by Shang, Marulic and Sturn[29]. The specimen dimensions and material used by Tsao and Ching are considered in the first group of numerical examples. The Young's modulus, E, for

Myler used by Tsao and Ching for the shell is not explicitly given in the papers. However, there are indications that the average values are $0.654X10^6$ psi and 10^6 psi from different sources, so that an intermediate value of $0.8X10^6$ psi is used in this numerical calculation. Other geometrical dimensions and material constants considered are as follows:

Radius of shell, $a = 4$ inches;

Length of shell, $\ell = 12$ inches;

Thickness of shells, $h = 0.01$ inches;

Poisson's ratio for shells, $\nu = 0.3$ (mylar);

Young's modulus of stringers, $E_s = 0.426X10^6$ psi (plexiglass).

The other properties of the stringers are given in Table 1. The stringers were bound to the inside of the shell along the circumference with their 0.25 inch dimension oriented along the circumferential direction of the shell for the tests. The results are given in Table 2. Since the ends of the cylinders were fixed to the end plates in testing, the boundary conditions are not the same as those in this analysis. However, it is interesting to note that the results compare favorably for the cylinder having six and nine stringers but unfavorably for the case where twelve and eighteen stringers are considered. This may indicate that the effects of end conditions are significant for certain shell-stringer combinations but less significant for other configurations. It is found that values of m for which σ_m is lowest are either one or two for this particular example. (See Table 2) The calculated buckling modes in the circumferential direction are plotted in Figure 5 for the case of a shell having 0.25"X 0.10" stringers. It is shown that all stringers are bent outward (or inward) for the six and nine stringer cases, while the stringers are bent inward and outward alternately for the twelve and eighteen stringers case. The critical loads according to the "smeared" analysis without considering the "effective" width of skin are also presented in Table 2, where it is shown that the "smeared" results are higher than the present results.

It is to be noted that the critical loads for the cases of twelve stringers are lower than the loads for a shell having only nine stringers. The test results given by Shang, Marulic and Sturn[29] similarly show that increasing the number of stringers of the same size reduces the critical loads for certain cases. This phenomenon may be closely related to the mode of buckling.

In the previous cases, it is apparent that all stringers are quite flexible. As a result , they all bend during buckling. Another example is now presented of a cylinder stiffened with stringers having large bending rigidity, with anticipated buckling of the skin only. The parameters used in the calculation are

$$\frac{a}{h} = 402 \ ; \quad \frac{\ell}{a} = 3 \ ; \quad \nu = 0.3 \ ; \quad \frac{E_s}{E} = 1 \ ;$$

$$\frac{A_s}{ah} = 0.3 \ ; \quad \frac{I_y}{ah^3} = 100 \ ; \quad \frac{J}{ah^3} = 50 \ ; \quad \frac{I_z}{a^3 h} = 0.05 \ ;$$

$$h_z = -0.005 \text{ (inside)} \ ; \quad y_o = z_o = h_y = C_1 = 0 \ .$$

The critical stress according to the present analysis is

$$\frac{p_{cr}}{E} = 0.001676 \ ,$$

while the classical critical stress for the simple supported panel is

$$\frac{p_{cr}}{E} = 0.001506 \ .$$

The number of half waves in the longitudinal direction is six and the mode shape in the circumferential direction is shown in Figure 6. It is seen that the stringers are twisted but not bent.

As the third example, the critical loads vs. the curvature parameter, Z, are plotted on a log-log scale as shown in Figure 7 for the case of a shell having six, ten and twelve equally spaced stringers for inside stiffening. Hutchinson and Amazigo[46] presented such results according to a "smeared" analysis. The nondimensionalized quantities used by Hutchinson and Amazigo are considered here. In these cases, the total area and total bending stiffness of the stringer are kept constant such that

Total stringer area = 12.6 (ah);

Total stringer bending stiffness, I_y, = 576 (ah^3);

Eccentricity, h_z, = -0.0375 (a) (inside),

and the critical loads are calculated for various lengths with $\frac{a}{h} = 400$ and $E_s = E$. The ordinate of the plots is the ratio of the total critical load of the stiffened cylindrical shell to that of the unstiffened monocoque thin cylindrical shell calculated according to the classical buckling load for simply supported cylinder, i.e.,

$$(P_{cr})_{\text{UNSTIFF.}} = 2\pi ah \frac{Eh}{a\sqrt{3(1-\nu^2)}}$$

It is to be noted that the critical loads are not sharply increasing to large value as the value Z decreases; this is unlike Hutchinson's results which are calculated according to a smeared analysis. The arrangement of the stringers greatly influences the critical loads.

It should be pointed out that if one does consider the longitudinal displacement of contacting point of the stringers as the average extension (or contraction) of the cross-section - i.e., one lets $h_z^i = h_y^i = 0$ in Eq.(2-55) - the critical loads change significantly for certain cases. For example, the critical loads are greatly reduced for the same case as given in Figure 7, as shown in Figure 8. Table 3 and Figure 9 show other examples of different results given by the approximation.

The eccentricity effects for a shell with six and twelve stringers are shown in Figure 9. The dimensionless parameters, $(P_{cr})/(P_{cr})_o$ and h_z/h, are used in the figure where (P_{cr}) is the total critical load, $(P_{cr})_o$ is the corresponding total critical load for the same shell with zero eccentricity, h_z is the eccentricity and h is the skin thickness. Other parameters used in these computations are

$$\frac{a}{h} = 400 \ ; \quad \frac{\ell}{a} = 3 \ ; \quad \nu = 0.3 \ ; \quad \frac{E_s}{E} = 1 \ ;$$

$$\frac{A_s}{ah} = 0.625 \ ; \quad \frac{I_y}{ah^3} = 32.55 \ ; \quad \frac{J}{ah^3} = 15.59 \ ;$$

$$\frac{I_z}{a^3 h} = 0.0000205 \ ; \quad y_o = z_o = h_y = 0 \ .$$

As a result, the total critical load for shell without eccentricity are

$$(P_{cr})_o = 0.01078 \ (ahE) \qquad \text{for 6 stringers case,}$$

and

$$(P_{cr})_o = 0.01523 \ (ahE) \qquad \text{for 12 stringers case,}$$

and the corresponding critical stresses $(p_{cr})_o$ are

$$\frac{(p_{cr})_o}{E} = 1.0744 \times 10^{-3} \quad \text{for 6 stringers case,}$$

and

$$\frac{(p_{cr})_o}{E} = 1.1045 \times 10^{-3} \quad \text{for 12 stringers case,}$$

where E is Young's modulus of the skin.

It is noted that the shape of the curves are similar for both configurations. The nondimensional critical load $(P_{cr})/(P_{cr})_o$ for the twelve stringer cases are less than those of the six stringer cases. However the actual total critical load P_{cr} for the former cases are higher than the latter cases. For a shell having outside stringers, the critical load increases as the eccentricity increases up to a certain value of eccentricity ($\frac{h_z}{h}$ = 10 for the case under study); then the critical load decreases as the eccentricity increases up to a certain value of the eccentricity ($\frac{h_z}{h}$ = 10); then the critical load increases as the eccentricity increases. Singer, Baruch and Harari[36] made similar investigations up to $\frac{h_z}{h}$ = 10 for different geometric parameters and obtained, for long stiffened cylinders, similar results. They indicated that the critical load increases for shells having outside stringers and decreases for shells with inside stringers as the eccentricity increases. Within the range of eccentricity ($0 \le \frac{h_z}{h} \le 15$) considered, it is interesting to note that the dashed lines are almost straight, and the corresponding critical loads are generally less than those of solid lines for outside stringer cases. The dashed lines intersect the solid lines at $\frac{h_z}{h} = 10$ for inside stringer cases.

The parameters $\frac{E_s A_s}{Ehad}$ and $\frac{E_s I_y}{Dad}$ are used in the smeared analysis to characterize whether or not the stiffening is light, medium or heavy. In Hutchinson's[46] work, the shell configuration parameters $\frac{E_s A_s}{Ehad} = 0.5$ and $\frac{E_s I_y}{Dad} = 20$, $\frac{E_s A_s}{Ehad} = 1$ and $\frac{E_s I_y}{Dad} = 100$, and $\frac{E_s A_s}{Ehad} = 2$ and $\frac{E_s I_y}{Dad} = 1000$ are identified as light, medium and heavy stiffening, respectively. The configurations used in the numerical examples cover the general ranges for light, medium and heavy stiffening with a maximum of eighteen stringers and a minimum of six stringers. The present results are all considerably lower than the results obtained according to the smeared analysis when the effective width of the skin is not considered. Since the numerical results made according to the present analysis all deviate from the results based on smeared analysis, the method presented should be used for longitudinally stiffened shells with number of stringers under eighteen.

In the previous examples, the rectangular stringers are used so that the warping rigidities are zero. If the stringers' cross-sections are channel or "I" section, the warping rigidities are no longer zero. To see the effect of the warping rigidity, the following example is considered:

$$\frac{a}{h} = 400 \ ; \quad \frac{\ell}{a} = 3 \ ; \quad \nu = 0.3 \ ; \quad \frac{E_s}{E} = 1 \ ;$$

$$\frac{A_s}{ah} = 0.703 \ ; \quad \frac{I_y}{ah^3} = 217.407 \ ; \quad \frac{I_z}{a^3 h} = 0.0001376 \ ;$$

$$\frac{J}{ah^3} = 1.953 \ ; \quad \frac{C_1}{a^3 h^3} = 0.1206875 \ ;$$

$$\frac{h_z}{a} = -0.04 \text{ or } -0.012 \text{ (inside)} \;; \quad h_y = y_o = z_o = I_{yz} = 0 \;.$$

For the cylindrical shell stiffened by six stringers, the critical stresses are

$$\frac{p_{cr}}{E} = 0.001510 \quad \text{for} \quad h_z = -0.4 \;,$$

and

$$\frac{p_{cr}}{E} = 0.001652 \quad \text{for} \quad h_z = -0.012.$$

If the warping rigidity is neglected ($C_1 = 0$), the critical stresses are

$$\frac{p_{cr}}{E} = 0.001472 \quad \text{for} \quad h_z = -0.04 \;,$$

and

$$\frac{p_{cr}}{E} = 0.001615 \quad \text{for} \quad h_z = -0.012.$$

It is noted that the critical stresses are respectively reduced by 2.54% and 2.15% if the warping rigidity is neglected.

Pre-Buckling Deformation Effects

A trial-and-error method is utilized for the numerical calculation in this analysis to find the lowest eigenvalue of Eq.(4-48). For each trial buckling load, the pre-buckling deformations are first calculated according to the derivation presented in Chapter IV. $m' = 1, 3, \ldots,$ 19 and $n' = 1, 2, 3, \ldots, 11$ are used in the series solution (Eq.(4-48)) for this set of numerical examples and the convergence of the solution is witness. The computer time required to obtain a solution according to the Galerkin method is formidable if the infinite series representation of displacement given in Eq.(4-44) is used. In view of the fact that the buckling mode, according to the linear analysis presented in Chapter III, is found to be represented by a single function, and the final buckling deformation according to the analysis presented in Chapter IV should not differ appreciably from that of the linear analysis, only a one-term approximation is considered in the Galerkin method instead of a longer finite series, i.e., Eq.(4-44) becomes

$$w'^{i}(\xi,\eta) = B_{mn} W'^{i}(\eta) \sin\frac{m\pi\xi}{L} \quad \text{for} \quad n = 1 \text{ and variable } m.$$

Note that here n = 1 corresponds to the ϵ_{mn} which is the lowest eigenvalue for each m. As a result, Eq.(4-46) becomes

$$K_{m1m1} B_{m1} = 0 \qquad \text{for each } m, \tag{5-2}$$

or

$$K_{m1m1} (\sigma, m) = 0 \quad \text{for each } m. \tag{5-3}$$

A trial-and-error method is used to determine the smallest value of σ for each m, say σ_m. Then, as in the linear analysis, the lowest σ_m will represent the eigenvalue corresponding to the critical load.

For the case of the internally stiffened cylindrical shell with six stringers of rectangular cross-section and properties

$$\frac{a}{h} = 400 \; ; \quad \frac{\ell}{a} = 3 \; ; \quad \frac{E_s}{E} = 0.533 \; ; \quad \nu = 0.3 \; ,$$

the ratio of the critical load considering the pre-buckling deformation effects to that of the linear analysis are shown in Table 4. It is noted that all ratios are less than unity.

In this example, numerical calculation of the pre-buckling deformation shows that the pre-buckling radial displacement, w_o, is almost constant except in the neighborhood of both edges and along the stringers. This suggests that the terms containing stress resultant increments in Eq.(4-46) may be negligibly small, and hence the term $D_{k\ell mnm'n'}$ in Eq.(4-48) is neglected to reduce the computation time in this example.

Table 1. Properties of Stringers

Size (Width X Depth)	Area		I_y		I_z		J	
	in^2	$\bar{A}^i = \frac{A}{ah}$	in^4	$\bar{I}_y = \frac{I_y}{ah^3}$	in^4	$\bar{I}_z = \frac{I_z}{a^3h}$	in^4	$\bar{J} = \frac{J}{ah^3}$
0.25"X0.20"	0.050	1.256	1.66×10^{-4}	42.128	2.60×10^{-4}	4.089×10^{-4}	3.421×10^{-4}	86.816
0.25"X0.15"	0.375	0.942	0.703×10^{-4}	17.844	1.95×10^{-4}	3.069×10^{-4}	1.760×10^{-4}	44.666
0.25"X0.10"	0.025	0.628	0.208×10^{-4}	5.287	1.302×10^{-4}	2.045×10^{-4}	0.624×10^{-4}	15.830

Table 2. Total Critical Loads in Pounds

Number of Stringer	Stringer Size Depth × Width	Present Analysis			Experimental Results**		Smeared Analysis***	
		Critical Stresses*	Total Critical Loads	m	Panel Buckling	General Buckling	$E_s = 0.8\times10^6$	$E_s = 0.654\times10^6$
6	0.10"x0.25"	770.5	255	1	258^{+9}_{-6}	258^{+9}_{-6}	457	400
	0.15"x0.25"	650.1	242	1	297^{+20}_{-27}	301^{+13}_{-8}	705	634
	0.20"x0.25"	610.9	251	1	298^{+26}_{-13}	457^{+18}_{-18}	964	892
9	0.10"x0.25"	706.9	262	2	297^{+3}_{-3}	297^{+3}_{-3}	532	477
	0.15"x0.25"	808.6	349	2	347^{+11}_{-11}	365^{+4}_{-5}	861	789
	0.20"x0.25"	860.0	423	1	410^{+8}_{-9}	568^{+8}_{-8}	1249	1176

* Classical panel buckling stress = 1210 psi.

** Shell speciments are fixed at both ends.

*** According to Singer[35] and Simitses[39] analyses.

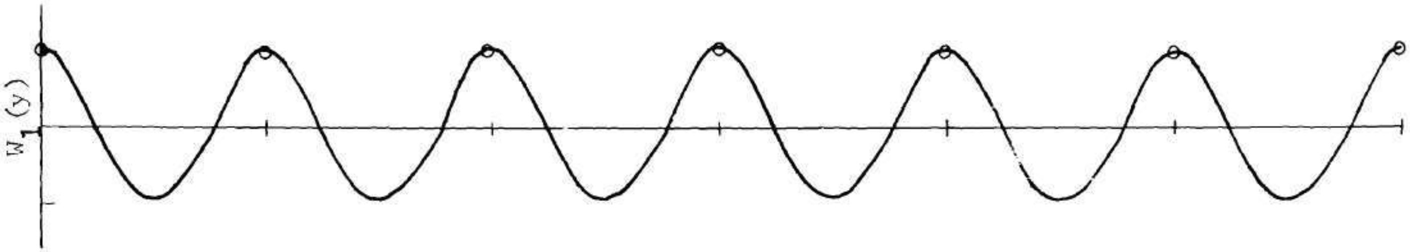
Table 2. Total Critical Loads in Pounds (Continued)

Number of Stringer	Stringer Size Depth x Width	Present Analysis			Experimental Results**		Smeared Analysis***	
		Critical Stresses*	Total Critical Loads	m	Panel Buckling	General Buckling	$E_s = 0.8\times10^6$	$E_s = 0.654\times10^6$
12	0.10"x0.25"	624.7	257	1	332^{+16}_{-20}	332^{+16}_{-20}	609	554
	0.15"x0.25"	497.9	244	1	419^{+31}_{-33}	437^{+8}_{-8}	1016	943
	0.20"x0.25"	467.9	267	1	480^{+10}_{-10}	798^{+5}_{-6}	1533	1460
18	0.10"x0.25"	544.9	268	2	369^{+3}_{-3}	489^{+11}_{-11}	762	706
	0.15"x0.25"	732.4	447	2	730^{+20}_{-23}	823^{+8}_{-9}	1324	1250
	0.20"x0.25"	866.5	663	2	964^{+18}_{-24}	1280^{+100}_{-88}	2100	1984

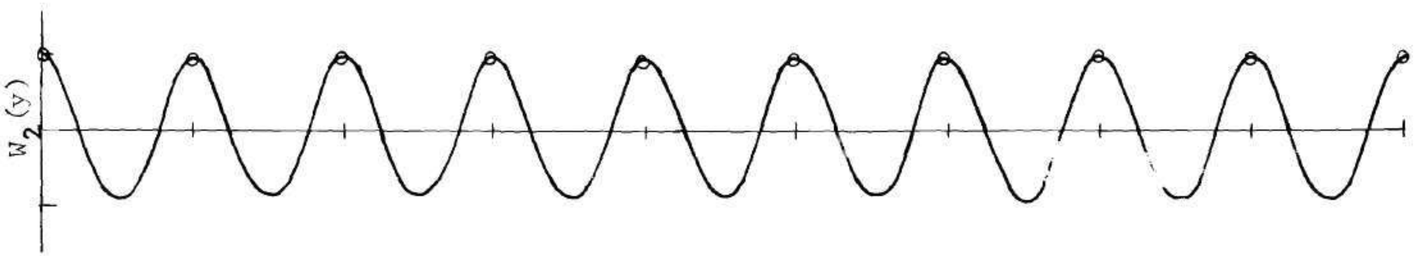
* Classical panel buckling stress = 1210 psi.

** Shell speciments are fixed at both ends.

*** According to Singer[35] and Simitses[39] analyses.

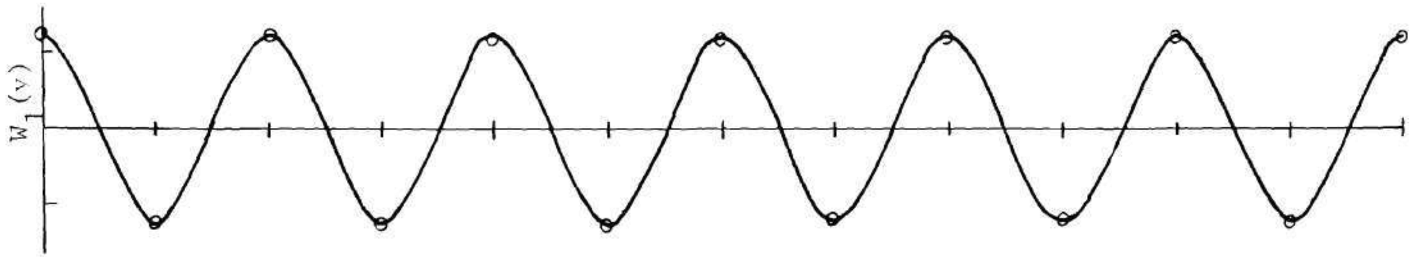


6 stringers: $w_{(x,y)} = W_1(y)\mathrm{Sin}\frac{\pi x}{\ell}$

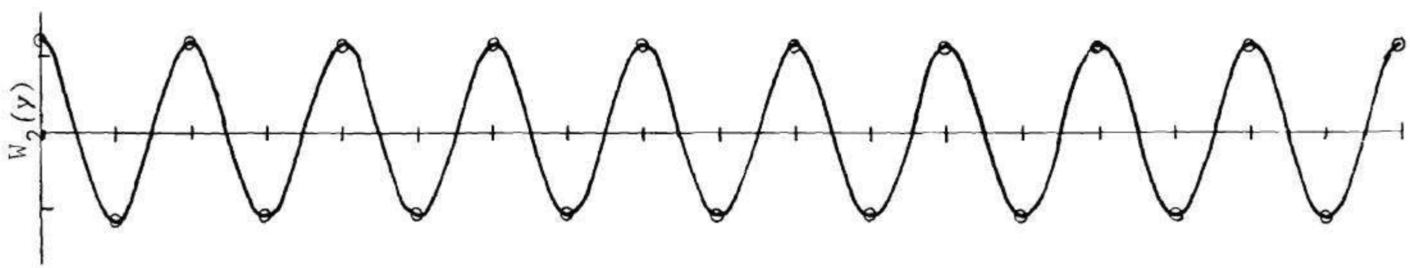


9 stringers: $w_{(x,y)} = W_2(y)\mathrm{Sin}\frac{2\pi x}{\ell}$

Figure 5. Buckling Mode of a Shell Having 0.10"×0.25" Stringer

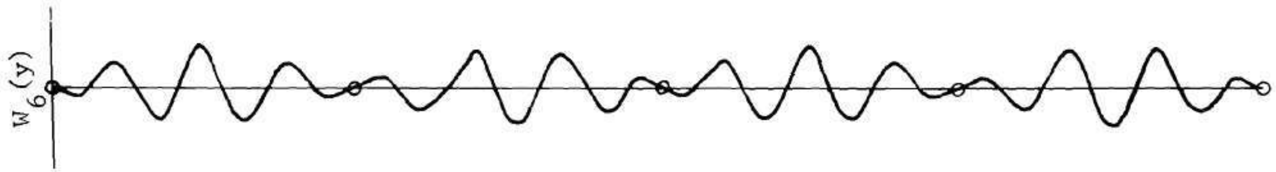


12 stringers : $w_{(x,y)} = W_1(y) \, \mathrm{Sin}\frac{\pi x}{\ell}$



18 stringers : $w_{(x,y)} = W_2(y) \, \mathrm{Sin}\frac{2\pi x}{\ell}$

Figure 5. (continue)



4 Stringers: $w(x, y) = W_6(y) \sin \frac{6\pi x}{\ell}$

Figure 6. Buckling Mode of the Second Example Shell

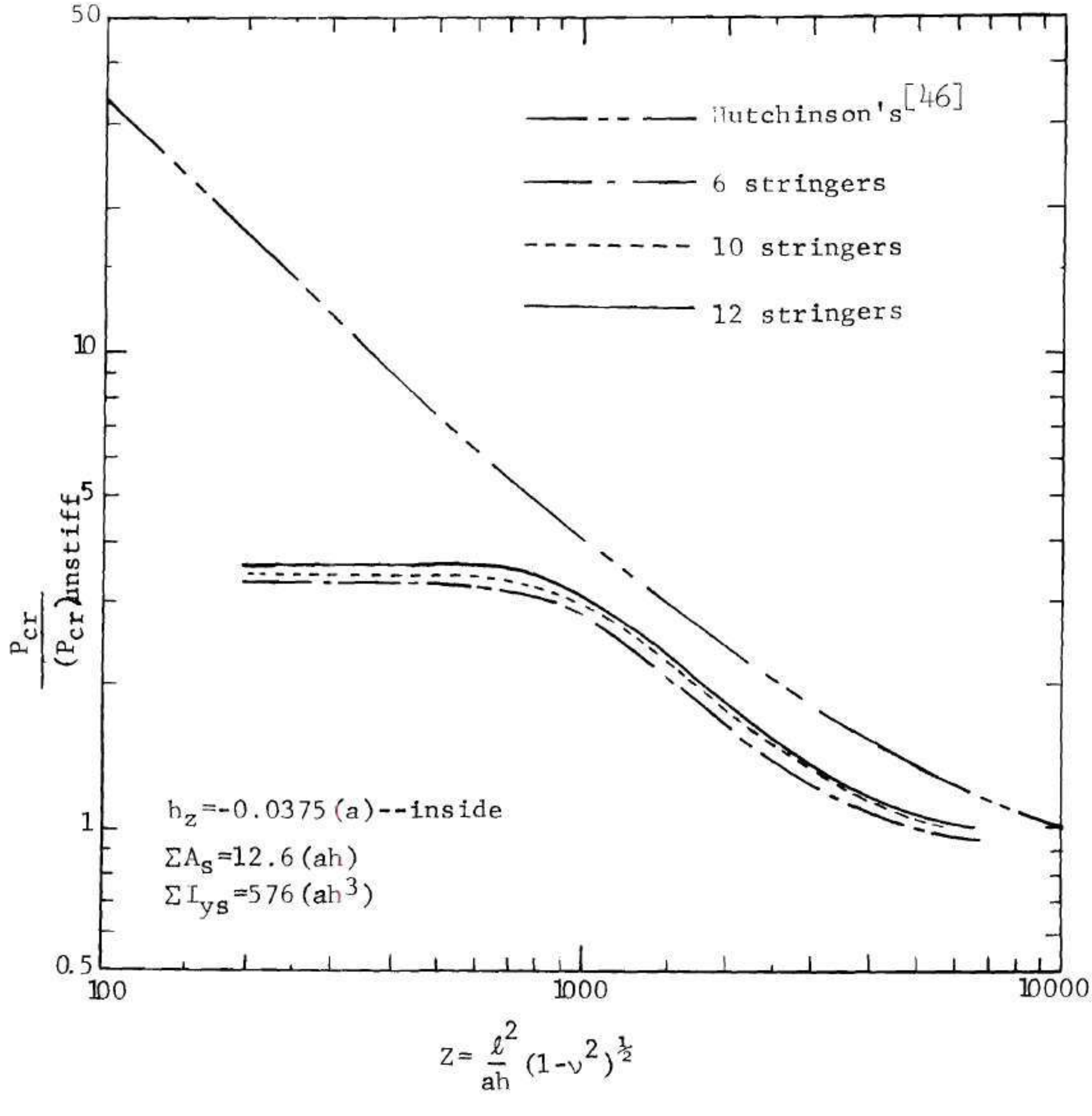


Figure 7. Critical Loads vs. Curvature Parameter

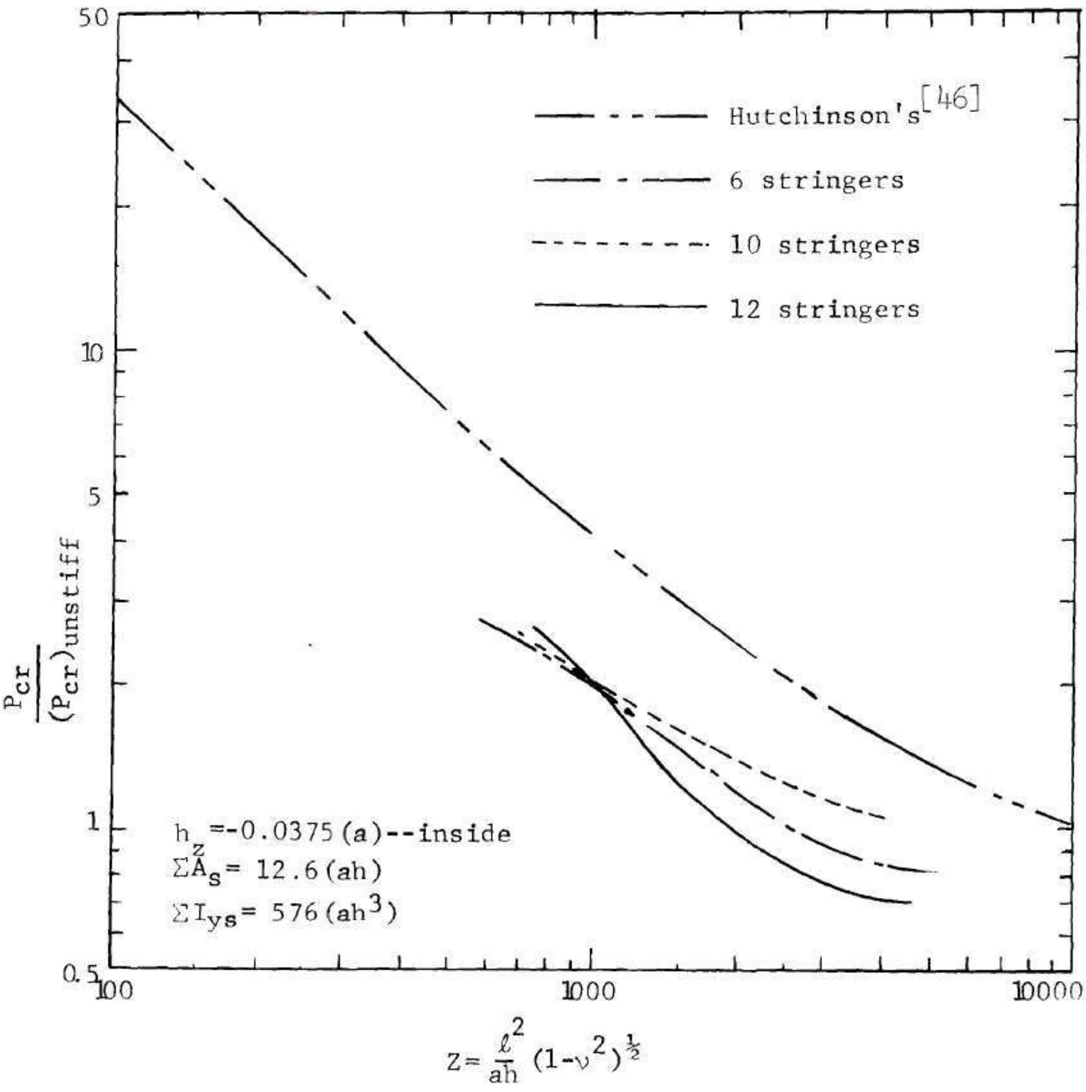


Figure 8. Critical Loads vs. Curvature Parameter
With Approximation in Longitudinal Contraction

Table 3. Comparison of Total Critical Load[a]

Number of Stringer	Stringer[b] Size (Depth X Width)	$(P_{cr})_A$[c] (Pounds)	$(P_{cr})_B$[d] (Pounds)
6	0.10"x0.25"	255	255
	0.15"x0.25"	242	252
	0.20"x0.25"	251	248
9	0.10"x0.25"	262	249
	0.15"x0.25"	349	278
	0.20"x0.25"	423	368
12	0.10"x0.25"	257	287
	0.15"x0.25"	244	274
	0.20"x0.25"	267	270
18	0.10"x0.25"	268	250
	0.15"x0.25"	447	290
	0.20"x0.25"	633	418

(a) Geometry and material constants used are

$$a = 4'' \ , \quad \ell = 12'' \ , \quad h = 0.01'' \ ,$$

$$E = 0.8\times10^{6} \text{ psi} \ , \quad \nu = 0.3 \ , \quad E_s = 0.426\times10^{6} \ .$$

(b) Inside stiffening.

(c) $(P_{cr})_A$ = total critical load given in Table 2.

(d) $(P_{cr})_B$ = total critical load with approximation in longitudinal contraction.

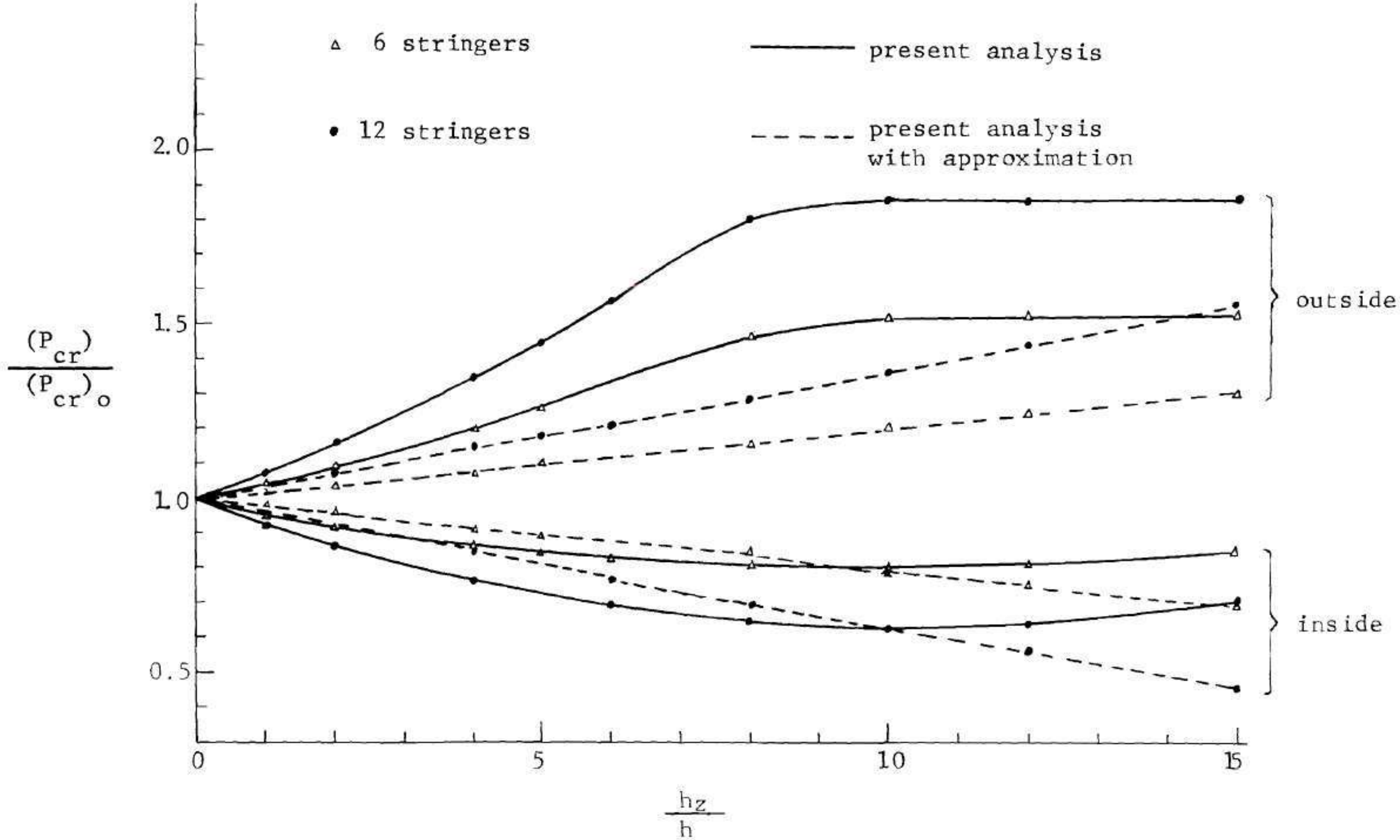


Figure 9. Critical Loads vs. Eccentricity

Table 4. Results of Pre-buckling Deformation Effects

Stringer Properties					$\frac{(P_{cr})}{(P_{cr})_{linear}}$
$\frac{A_s}{ah}$	$\frac{I_y}{ah^3}$	$\frac{J}{ah^3}$	$\frac{I_z}{a^3h}$	$\frac{h_z}{a}$	
0.628	5.29	15.82	0.0002	- 0.0125	0.89
0.942	17.84	44.67	0.0003	- 0.0187	0.86
1.256	42.13	86.82	0.0004	- 0.025	0.92

APPENDIX A

DERIVATION OF DONNELL TYPE EQUATION

For the cylindrical thin shell, if lines perpendicular to the middle surface remain so during distortion, then the displacement of all points in the cylindrical wall can be found from the displacement components of the middle surface, u,v and w. If the coordinate axes are set in such a way that the x-coordinate is in the longitudinal direction and the y-coordinate in the circumferential direction, according to the right-hand rule as shown in Figure 1, the equations of equilibrium for large deformation are

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} + q_x = 0 \,, \tag{A-1}$$

$$\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} + q_y - \frac{Q_y}{a} = 0 \,, \tag{A-2}$$

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + q + \frac{N_y}{a} + \frac{\partial}{\partial x}\left(N_x \frac{\partial w}{\partial x}\right) + \frac{\partial}{\partial x}\left(N_{xy} \frac{\partial w}{\partial y}\right) \tag{A-3}$$

$$+ \frac{\partial}{\partial y}\left(N_{xy} \frac{\partial w}{\partial x}\right) + \frac{\partial}{\partial y}\left(N_y \frac{\partial w}{\partial y}\right) = 0 \,,$$

$$\frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_x = 0 \,, \tag{A-4}$$

$$\frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} - Q_y = 0 \,, \tag{A-5}$$

where the axial deflection w is positive when deflecting inward, and $\bar{q} : (q_x, q_y, q)$ is the external surface load.

The strains, changes of curvature, and displacement components are related by the expressions

$$\epsilon_x = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 , \qquad \epsilon_y = \frac{\partial v}{\partial y} - \frac{w}{a} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 , \tag{A-6}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y} ,$$

$$K_x = \frac{\partial^2 w}{\partial x^2} , \qquad K_y = \frac{1}{a}\frac{\partial v}{\partial y} + \frac{\partial^2 w}{\partial y^2} , \qquad k_{xy} = \frac{1}{a}\frac{\partial v}{\partial x} + \frac{\partial^2 w}{\partial x \partial y} . \tag{A-7}$$

According to the Donnell's approximations[12], first, in connection with the equations of equilibrium, the transverse shearing force, Q_y, is considered to have negligible contribution to the equilibrium of the forces in the circumferential direction. This assumption can be expected to improve in accuracy as the ratio of the radius to the thickness of the shell increases. Therefore, the term Q_y/a in Eq.(A-2) can be neglected and, after eliminating Q_x and Q_y by means of Eqs.(A-4) and (A-5), the equations of equilibrium for the cylindrical shell are reduced to the set

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} + q_x = 0 , \tag{A-8}$$

$$\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} + q_y = 0 , \tag{A-9}$$

$$\frac{\partial^2 M_x}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} + q + \frac{N_y}{a} + \frac{\partial}{\partial x}\left(N_x \frac{\partial w}{\partial x}\right) \tag{A-10}$$

$$+ \frac{\partial}{\partial x}\left(N_{xy}\frac{\partial w}{\partial y}\right) + \frac{\partial}{\partial y}\left(N_{xy}\frac{\partial w}{\partial x}\right) + \frac{\partial}{\partial y}\left(N_y \frac{\partial w}{\partial y}\right) = 0 .$$

Secondly, in connection with the relationships between the changes of curvature, twist, and displacement, the magnitude of the changes of curvature and twist are negligibly affected by the "stretching" displacement component, v. Consequently, it is assumed that the changes in curvature and twist of a thin cylindrical shell, Eq.(A-7), can be expressed as

$$\kappa_x = \frac{\partial^2 w}{\partial x^2}, \quad \kappa_y = \frac{\partial^2 w}{\partial y}, \quad \kappa_{xy} = \frac{\partial^2 w}{\partial x \partial y}, \tag{A-11}$$

which are identical to the corresponding equation of the theory of thin plates.

Since the stress resultant-strain relations and the stress couple-curvature relations are

$$N_x = \frac{Eh}{1-\nu^2}(\epsilon_x + \nu\epsilon_y), \quad N_y = \frac{Eh}{1-\nu^2}(\epsilon_y + \nu\epsilon_x), \quad N_{xy} = Gh\gamma_{xy}, \tag{A-12}$$

and

$$M_x = -D(\kappa_x + \nu\kappa_y), \quad M_y = -D(\kappa_y + \nu\kappa_x), \quad M_{xy} = -\frac{D}{2}(1-\nu)\kappa_{xy}, \tag{A-13}$$

the relationships between stress resultants and the displacements and between the stress couples and the displacements can be expressed as

$$N_x = \frac{Eh}{1-\nu^2}\left[\frac{\partial u}{\partial x} + \nu\left(\frac{\partial v}{\partial y} - \frac{w}{a}\right) + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 + \frac{\nu}{2}\left(\frac{\partial w}{\partial y}\right)^2\right], \tag{A-14}$$

$$N_y = \frac{Eh}{1-\nu^2}\left[\frac{\partial v}{\partial y} - \frac{w}{a} + \nu\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 + \frac{\nu}{2}\left(\frac{\partial w}{\partial x}\right)^2\right], \tag{A-15}$$

$$N_{xy} = \frac{Eh}{2(1-\nu)}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right), \tag{A-16}$$

$$M_x = -D\left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2}\right), \tag{A-17}$$

$$M_y = -D\left(\frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2}\right), \tag{A-18}$$

$$M_{xy} = -D(1-\nu)\frac{\partial^2 w}{\partial x \partial y} . \tag{A-19}$$

The compatibility condition is

$$\frac{\partial^2 \epsilon_x}{\partial y^2} + \frac{\partial^2 \epsilon_y}{\partial x^2} - \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - \frac{1}{a}\frac{\partial^2 w}{\partial x^2} . \tag{A-20}$$

By using the stress resultant-strain relationship and equations of equilibrium, Eqs.(A-8) and (A-9), Eq.(A-20) can be rewritten as

$$\frac{1}{Eh}\left[\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(N_x + N_y) + (1+\nu)\left(\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y}\right)\right] \tag{A-21}$$

$$= \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - \frac{1}{a}\frac{\partial^2 w}{\partial x^2} .$$

If the stress resultants are expressed in terms of stress function, F, such that

$$N_x = \frac{\partial^2 F}{\partial y^2} - \int q_x \, dx , \quad N_y = \frac{\partial^2 F}{\partial x^2} - \int q_y \, dy , \quad N_{xy} = -\frac{\partial^2 F}{\partial x \partial y} , \tag{A-22}$$

then the equations of equilibrium,Eqs.(A-8) and (A-9) are satisfied. The compatibility condition, Eq.(A-21), can be expressed as

$$\frac{1}{Eh}\left[\nabla^4 F + \nu\left(\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y}\right) - \left(\frac{\partial^2}{\partial x^2}\int q_y \, dy + \frac{\partial^2}{\partial y^2}\int q_x \, dx\right)\right] \tag{A-23}$$

$$= \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - \frac{1}{a}\frac{\partial^2 w}{\partial x^2} .$$

Substituting Eqs.(A-17), (A-18) and (A-19) as well as the stress function, F, into Eq.(A-10), one obtains

$$D\nabla^4 w = q + \frac{1}{a}\left(\frac{\partial^2 F}{\partial x^2} - \int q_y\,dy\right) + \frac{\partial}{\partial x}\left[\left(\frac{\partial^2 F}{\partial y^2} - \int q_x\,dx\right)\frac{\partial w}{\partial x}\right] \tag{A-24}$$

$$-\frac{\partial}{\partial x}\left(\frac{\partial^2 F}{\partial x\partial y}\frac{\partial w}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{\partial^2 F}{\partial x\partial y}\frac{\partial w}{\partial x}\right) + \frac{\partial}{\partial y}\left[\left(\frac{\partial^2 F}{\partial x^2} - \int q_y\,dy\right)\frac{\partial w}{\partial y}\right].$$

Eqs.(A-23) and (A-24) should be solved simultaneously.

To get the relationships between the displacement components, u, v and w, one may substitute Eqs.(A-14), (A-15) and (A-16) into Eqs.(A-8) and (A-9), and obtain

$$\frac{\partial^2 u}{\partial x^2} + \frac{1-\nu}{2}\frac{\partial^2 u}{\partial y^2} + \frac{1+\nu}{2}\frac{\partial^2 v}{\partial x\partial y} - \frac{\nu}{a}\frac{\partial w}{\partial x} = -\frac{(1-\nu^2)}{Eh}\,q_x\,, \tag{A-25}$$

$$\frac{1-\nu}{2}\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{1+\nu}{2}\frac{\partial^2 u}{\partial x\partial y} - \frac{1}{a}\frac{\partial w}{\partial y} = -\frac{(1-\nu^2)}{Eh}\,q_y\,, \tag{A-26}$$

by neglecting the nonlinear terms. Rearrangment of Eqs.(A-25) and (A-26) results in

$$\frac{\partial^2 v}{\partial x\partial y} = -\frac{2}{1+\nu}\left(\frac{\partial^2 u}{\partial x^2} + \frac{1-\nu}{2}\frac{\partial^2 u}{\partial y^2} - \frac{\nu}{a}\frac{\partial w}{\partial x} + \frac{1-\nu^2}{Eh}q_x\right), \tag{A-27}$$

$$\frac{\partial^2 u}{\partial x\partial y} = -\frac{2}{1+\nu}\left(\frac{\partial^2 v}{\partial y^2} + \frac{1-\nu}{2}\frac{\partial^2 v}{\partial x^2} - \frac{1}{a}\frac{\partial w}{\partial y} + \frac{1-\nu^2}{Eh}q_y\right). \tag{A-28}$$

Now operating on Eqs.(A-26) and (A-25) with $\frac{\partial^2}{\partial x\partial y}$ yields

$$\frac{1-\nu}{2}\frac{\partial^2}{\partial x^2}\frac{\partial^2 v}{\partial x\partial y} + \frac{\partial^2}{\partial y^2}\frac{\partial^2 v}{\partial x\partial y} + \frac{1+\nu}{2}\frac{\partial^4 u}{\partial x^2\partial y^2} - \frac{1}{a}\frac{\partial^3 w}{\partial x\partial y^2} = -\frac{1-\nu^2}{Eh}\frac{\partial^2 q_y}{\partial x\partial y}\,, \tag{A-29}$$

$$\frac{\partial^2}{\partial x^2}\frac{\partial^2 u}{\partial x\partial y} + \frac{1-\nu}{2}\frac{\partial^2}{\partial y^2}\frac{\partial^2 u}{\partial x\partial y} + \frac{1+\nu}{2}\frac{\partial^4 v}{\partial x^2\partial y^2} - \frac{\nu}{a}\frac{\partial^3 w}{\partial x^2\partial y} = -\frac{1-\nu^2}{Eh}\frac{\partial^2 q_x}{\partial x\partial y}\,. \tag{A-30}$$

Substituting Eqs.(A-27) and (A-28) into Eqs.(A-29) and (A-30), respectively, and rearranging, the displacement components, u and v, can be expressed in terms of the displacement component, w, as

$$\nabla^4 u = -\frac{1}{a}\frac{\partial^3 w}{\partial x \partial y^2} + \frac{\nu}{a}\frac{\partial^3 w}{\partial x^3} - \frac{1-\nu^2}{Eh}\left(\frac{\partial^2 q_x}{\partial x^2} + \frac{2}{1-\nu}\frac{\partial^2 q_x}{\partial y^2} - \frac{1+\nu}{1-\nu}\frac{\partial^2 q_y}{\partial x \partial y}\right), \qquad \text{(A-31)}$$

$$\nabla^4 v = \frac{2+\nu}{a}\frac{\partial^3 w}{\partial x^2 \partial y} + \frac{1}{a}\frac{\partial^3 w}{\partial y^3} - \frac{1-\nu^2}{Eh}\left(\frac{\partial^2 q_y}{\partial y^2} + \frac{2}{1-\nu}\frac{\partial^2 q_y}{\partial x^2} - \frac{1+\nu}{1-\nu}\frac{\partial^2 q_x}{\partial x \partial y}\right). \qquad \text{(A-32)}$$

For the purpose of studying the buckling problem, one can combine Eqs.(A-23) and (A-24) to obtain Chien's equation[61]:

$$D\nabla^8 w + \frac{Eh}{a^2}\frac{\partial^4 w}{\partial x^4} - \nabla^4\left[\frac{\partial}{\partial x}\left(N_x \frac{\partial w}{\partial x}\right) + \frac{\partial}{\partial x}\left(N_{xy}\frac{\partial w}{\partial y}\right) + \frac{\partial}{\partial y}\left(N_{xy}\frac{\partial w}{\partial x}\right)\right. \qquad \text{(A-33)}$$

$$\left. + \frac{\partial}{\partial y}\left(N_y \frac{\partial w}{\partial y}\right)\right] = \nabla^4 q + \frac{Eh}{a}\frac{\partial^2}{\partial x^2}\left[\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}\right]$$

$$- \frac{1}{a}\left[\nu\frac{\partial^3 q_x}{\partial x^3} - \frac{\partial^3 q_x}{\partial x \partial y^2} + \frac{\partial^3 q_y}{\partial y^3} + (2+\nu)\frac{\partial^3 q_y}{\partial x^2 \partial y}\right].$$

Let

$$w = w_o + w' \quad ; \qquad N_x = N_{ox} + N'_x \quad ; \qquad \text{(A-34)}$$

$$N_y = N_{oy} + N'_y \quad ; \quad N_{xy} = N_{oxy} + N'_{xy} \quad ,$$

where w_o, N_{ox}, N_{oy} and N_{oxy} are the radial displacement and stress resultants in the pre-buckling state, respectively, which satisfy Eq.(A-33), and w', N'_x, N'_y and N'_{xy} are buckling increments which are assumed to be small such that terms nonlinear in these quantities can be neglected compared to terms linear in these quantities. By substituting Eq.(A-34) into Eq.(A-33), noting that w_o, N_{ox}, N_{oy} and N_{oxy} satisfy Eq.(A-33), linearizing with respect to primed quantities, one may obtain the linearized differential equation governing the buckling increment, w':

$$D\nabla^8 w' + \frac{Eh}{a^2}\frac{\partial^4 w'}{\partial \xi^4} - \nabla^4 \Big[\frac{\partial}{\partial x}\Big(N_{ox}\frac{\partial w'}{\partial x}\Big) + \frac{\partial}{\partial x}\Big(N_{oxy}\frac{\partial w'}{\partial y}\Big) \tag{A-35}$$

$$+ \frac{\partial}{\partial y}\Big(N_{oxy}\frac{\partial w'}{\partial x}\Big) + \frac{\partial}{\partial y}\Big(N_{oy}\frac{\partial w'}{\partial y}\Big) + N'_x\frac{\partial^2 w_o}{\partial x^2} + 2N'_{xy}\frac{\partial^2 w_o}{\partial x\partial y} + N'_y\frac{\partial^2 w_o}{\partial y^2}\Big]$$

$$= \frac{Eh}{a}\frac{\partial^2}{\partial x^2}\Big(2\frac{\partial^2 w_o}{\partial x\partial y}\frac{\partial^2 w'}{\partial x\partial y} - \frac{\partial^2 w_o}{\partial x^2}\frac{\partial^2 w'}{\partial y^2} - \frac{\partial^2 w_o}{\partial y^2}\frac{\partial^2 w'}{\partial x^2}\Big).$$

APPENDIX B

GOVERNING DIFFERENTIAL EQUATIONS FOR OPEN THIN WALLED BEAM

The bending and torsion of a beam of open thin walled section that is elastically supported throughout its length by skin and subjected to an axial compressive force will be considered here. In addition to the axial load, external forces and moments are acting along the intersection lines of the skin and stringers due to the reactions between the skin and the stringers.

The analysis is based on the usual thin walled beam assumptions of undeformable section contour, and zero shear deformation along the middle surface of the beam.

Under the fundamental assumption that the section is undeformable in its plane, the only possible motion of the section in its own plane is the rigid body motion which can be described by three coordinates: two linear displacement components, v_s and w_s, of any reference point, and one angular displacement, ϕ_s. Since the torques applied at the free ends of the beam will produce no deflection of the shear center, the reference point is chosen as the shear center of the cross-section to simplify the algebra of the problem.

Let the beam be bent in x-z plane. The equations of equilibrium are

$$\frac{dV_0}{dx} = -V_{ys} + \frac{d}{dx}\left[N_0^* \frac{d}{dx}(w_s - y_s\phi_s)\right], \tag{B-1}$$

and

$$\frac{dM_o}{dx} = V_o - h_z N_{xys} \,, \tag{B-2}$$

as referred to Figure 10.

Since the bending equation of the beam about the y-axis is

$$E_s I_y \frac{d^2 w_s}{dx^2} + E_s I_{yz} \frac{d^2 v_s}{dx^2} = -M_o \,, \tag{B-3}$$

the elimination of V_o between Eqs.(B-1) and (B-2), the use of Eq.(B-3) yields

$$E_s I_y \frac{d^4 w_s}{dx^4} + E_s I_{yz} \frac{d^4 v_s}{dx^4} + \frac{d}{dx}\left[N_o^* \frac{d}{dx}(w_s - y_o \phi_s)\right] = V_{ys} + \frac{d}{dx}(h_z N_{xys}). \tag{B-4}$$

If N_o^* is considered to be constant, say N_o, and if h_z is constant, then Eq.(B-4) can be written as

$$E_s I_y \frac{d^4 w_s}{dx^4} + E_s I_{yz} \frac{d^4 v_s}{dx^4} + N_o \frac{d^2}{dx^2}(w_s - y_o \phi_s) = V_{ys} + h_z \frac{dN_{xys}}{dx}. \tag{B-5}$$

In exactly the same way, the differential equation governing the stringer bending in the x-y plane can be expressed as

$$E_s I_z \frac{d^4 v_s}{dx^4} + E_s I_{yz} \frac{d^4 w_s}{dx^4} + N_o \frac{d^2}{dx^2}(v_s + z_o \phi_s) = N_{ys} + h_y \frac{dN_{xys}}{dx}. \tag{B-6}$$

The differential equation governing the angle of twist, ϕ_s, of the beam of the thin walled open section is

$$C_1 \frac{d^4 \phi_s}{dx^4} - C \frac{d^2 \phi_s}{dx^2} = m_x + \int_C \omega(s) \frac{\partial P_x(s,x)}{\partial x} ds \,, \tag{B-7}$$

as given in [62]. In Eq.(B-7), $C_1 = E_s C_w$ is the warping rigidity and

$C = G_S J$ is the torsional rigidity, where the warping constant, C_w, and the torsional constant, J, are geometric constants as listed in Table A-3 of [9]. $\omega(s)$ is the sectorial area such that $\omega = \int_{s_0}^{s} r\,ds$ with the shear center as the pole. The starting point s_o is chosen such that $\int_A \omega_{(s)}\,dA = 0$ (see Figure 10). $p_{(s,x)}$ is the surface force in the longitudinal direction. In the present case,

$$p_{(s,x)} = \delta_{(s-s_N)} N_{xys}(x),$$

where $\delta_{(s)}$ is delta function and s_N locates the contact point of skin and stringer.

The intensity m_x of the torque distributed along the shear center axis, as shown in Figure 11, is equal to the couple developed by the load of the lateral forces obtained from the action of the initial compressive forces acting on the slightly rotated cross-sections of the longitudinal fibers, plus the external torque developed by the skin. The compressive forces acting on the slightly rotated ends of the element will give the forces in the y and z directions.

The displacements in y and z directions of any point in a section are $v_s + (z_o - z)\phi_s$ and $w_s - (y_o - y)\phi_s$ respectively. The components of the compressive forces $\sigma_x t ds$ in the y and z directions after deformation of the stringer are

$$-(\sigma_x t\,ds)\frac{d^2}{dx^2}\left[v_s + (z_o - z)\phi_s\right], \tag{B-8}$$

and

$$-(\sigma_x t\,ds)\frac{d^2}{dx^2}\left[w_s - (y_o - y)\phi_s\right]. \tag{B-9}$$

where t is the thickness of the cross-section, ds is the distance measured along the middle line, and σ_x is the axial compressive stress which is assumed independent of x.

By taking moment about the shear center of the forces given in the expressions (B-8) and (B-9), and considering all external forces and moments, one obtains

$$m_x = \int_c -\left\{ (z_o - z)\left[\frac{d^2 v_s}{dx^2} + (z_o - z)\frac{d^2\phi_s}{dx^2}\right] - (y_o - y)\left[\frac{d^2 w_s}{dx^2}\right.\right. \tag{B-10}$$

$$\left.\left. - (y_o - y)\frac{d^2\phi_s}{dx^2}\right]\right\}\sigma_x t\, ds + (z_o - h_z)N_{ys} - (y_o - h_y)V_{ys} - M_{ys}.$$

Observing that

$$\int_c \sigma_x t\, ds = N_o \ , \quad \int_c y t\, ds = \int_c z t\, ds = 0 \ , \quad \int_c y^2 t\, ds = I_z \ ,$$

$$\int_c z^2 t\, ds = I_y \ , \qquad I_o = I_y + I_z + A_s(y_o^2 + z_o^2).$$

Eq.(B-10) can be re-written as

$$m_x = N_o\left[y_o \frac{d^2 w_s}{dx^2} - z_o \frac{d^2 v_s}{dx^2}\right] - \frac{I_o}{A_s} N_o \frac{d^2\phi_s}{dx^2} + (z_o - h_z)N_{ys} \tag{B-11}$$

$$-(y_o - h_y)V_{ys} - M_{ys}.$$

After substitution of Eq.(B-11) into Eq.(B-7) the governing differential equation for non-uniform torsion becomes

$$-C_1\frac{d^4\phi_s}{dx^4} + \left(C - \frac{I_o}{A_s}N_o\right)\frac{d^2\phi_s}{dx^2} + N_o y_o \frac{d^2 w_s}{dx^2} - N_o z_o \frac{d^2 v_s}{dx^2} \tag{B-12}$$

$$= M_{ys} + (y_o - h_y)V_{ys} - (z_o - h_z)N_{ys} + \omega_{(s_N)}\frac{dN_{xys}}{dx}.$$

The fourth governing differential equation is obtained by considering the contraction of the stringer in the longitudinal direction. The equation of equilibrium in the x-direction is

$$\frac{dN_o^*}{dx} = N_{xys} \qquad \text{where} \qquad N_o^* = -\int_{A_s} \sigma_x^* \, dA \ , \tag{B-13}$$

and the stress-strain relation yields

$$\sigma_x^* = E_s \epsilon_x = E \left[\frac{du}{dx} + \frac{1}{2}\left(\frac{dw}{dx}\right)^2 + \frac{1}{2}\left(\frac{dv}{dx}\right)^2 \right] . \tag{B-14}$$

At any point (y, z) on the cross-section, the displacement components u, v and w can be expressed as

$$u = u_s - (z - z_o)\frac{dw_s}{dx} - (y - y_o)\frac{dv_s}{dx} - \omega_{(s)}\frac{d\phi_s}{dx} \ ,$$

$$v = v_s + (z_o - z)\,\phi_s \ ,$$

$$w = w_s - (y_o - y)\,\phi_s \ ,$$

where $\omega_{(s)}$ is the sectorial area such that $\omega_{(s)} = \int_{s_o}^{s} r\,ds$ (see Figure 10). Hence

$$\sigma_x^* = E\left\{ \frac{du_s}{dx} - (z - z_o)\frac{d^2w_s}{dx^2} - (y - y_o)\frac{d^2v_s}{dx^2} - \omega_{(s)}\frac{d^2\phi_s}{dx^2} \right.$$

$$\left. + \frac{1}{2}\left[\frac{dw_s}{dx} - (y - y_o)\frac{d\phi_s}{dx}\right]^2 + \frac{1}{2}\left[\frac{dv_s}{dx} + (z_o - z)\frac{d\phi_s}{dx}\right]^2 \right\} .$$

Since the contour point s_o is chosen to be such that

$$\int_{A_s} \omega_{(s)}\, dA = 0 \ ,$$

then

$$N_c^* = -\int_{A_s} \sigma_x^* \, dA = -E_s A_s \Big(\frac{du_s}{dx} + z_o \frac{d^2 w_s}{dx^2} + y_o \frac{d^2 v_s}{dx^2}$$
$$+ \frac{1}{2}\left(\frac{dw_s}{dx}\right)^2 + \frac{1}{2}\left(\frac{dv_s}{dx}\right)^2 - \left(y_o \frac{dw_s}{dx} - z_o \frac{dv_s}{dx}\right) + \frac{1}{2} I_o \left(\frac{d\phi_s}{dx}\right)^2 \Big) , \tag{B-15}$$

and

$$-E_s A_s \Big\{ \frac{d^2 v_s}{dx^2} + z_o \frac{d^3 w_s}{dx^3} + y_o \frac{d^3 v_s}{dx^3} + \frac{d}{dx} \Big[\frac{1}{2}\left(\frac{dw_s}{dx}\right)^2 + \frac{1}{2}\left(\frac{dv_s}{dx}\right)^2$$
$$+ \frac{1}{2} I_o \left(\frac{d\phi_s}{dx}\right)^2 - \left(y_o \frac{dw_s}{dx} - z_o \frac{dv_s}{dx}\right) \frac{d\phi_s}{dx} \Big] \Big\} = N_{xys} . \tag{B-16}$$

In the buckled state, let

$$u_s = u_{os} + u_s' , \quad v_s = v_{os} + v_s' , \quad w_s = w_{os} + w_s' , \tag{B-17}$$
$$\phi_s = \phi_{os} + \phi_s' , \quad N_{ys} = N_{oys} + N_{ys}' , \quad N_{xys} = N_{oxys} + N_{xys}' ,$$
$$V_{ys} = V_{oys} + V_{oys}' , \quad M_{ys} = M_{oys} + M_{ys}' ,$$

where the quantities with subscript "o" denote the pre-buckling state quantities which satisfy Eqs.(B-5), (B-6), (B-12) and (B-16), and the primed quantities are the buckling increments which are assumed to be small such that the terms nonlinear in these quantities can be neglected compared to terms linear in these quantities. Then the differential equations governing the buckling increments for the open section thin walled beam become

$$E_s I_y \frac{d^4 w_s'}{dx^4} + E_s I_{yz} \frac{d^4 v_s'}{dx^4} + N_o \frac{d^2}{dx^2}(w_s' - y_o \phi_s') = V_{ys}' + h_z \frac{d}{dx} N_{xys}' \tag{B-18}$$

$$E_s I_z \frac{d^4 v_s'}{dx^4} + E_s I_{yz} \frac{d^4 w_s'}{dx^4} + N_o \frac{d^2}{dx^2}(v_s' + z_o \Phi_s') = N_{ys}' + h_y \frac{d}{dx} N_{xys}' , \tag{B-19}$$

$$-C_1 \frac{d^4 \Phi_s'}{dx^4} + \left(C - \frac{I_o}{A_s} N_o\right) \frac{d^2 \Phi_s'}{dx^2} + N_o y_o \frac{d^2 w_s'}{dx^2} - N_o z_o \frac{d^2 v_s'}{dx^2} \tag{B-20}$$

$$= M_{ys}' + (y_o - h_y) V_{ys}' - (z_o - h_z) N_{ys}' + \omega_{(sN)} \frac{dN_{xys}'}{dx} ,$$

$$-E_s A_s \left\{ \frac{d^2 u_s}{dx^2} + z_o \frac{d^3 w_s'}{dx^3} + y_o \frac{d^3 v_s'}{dx^3} + \frac{d}{dx} \left[\frac{dw_{os}}{dx} \frac{dw_s'}{dx} + \frac{dv_{os}}{dx} \frac{dv_s'}{dx} \right.\right. \tag{B-21}$$

$$\left.\left. + I_o \frac{d\Phi_{os}}{dx} \frac{d\Phi_s'}{dx} - \left(y_o \frac{dw_s'}{dx} - z_o \frac{dv_s'}{dx} \right) \frac{d\Phi_{os}}{dx} - \left(y_o \frac{dw_{os}}{dx} - z_o \frac{dv_{os}}{dx} \right) \frac{d\Phi_s'}{dx} \right] \right\}$$

$$= N_{xys}' .$$

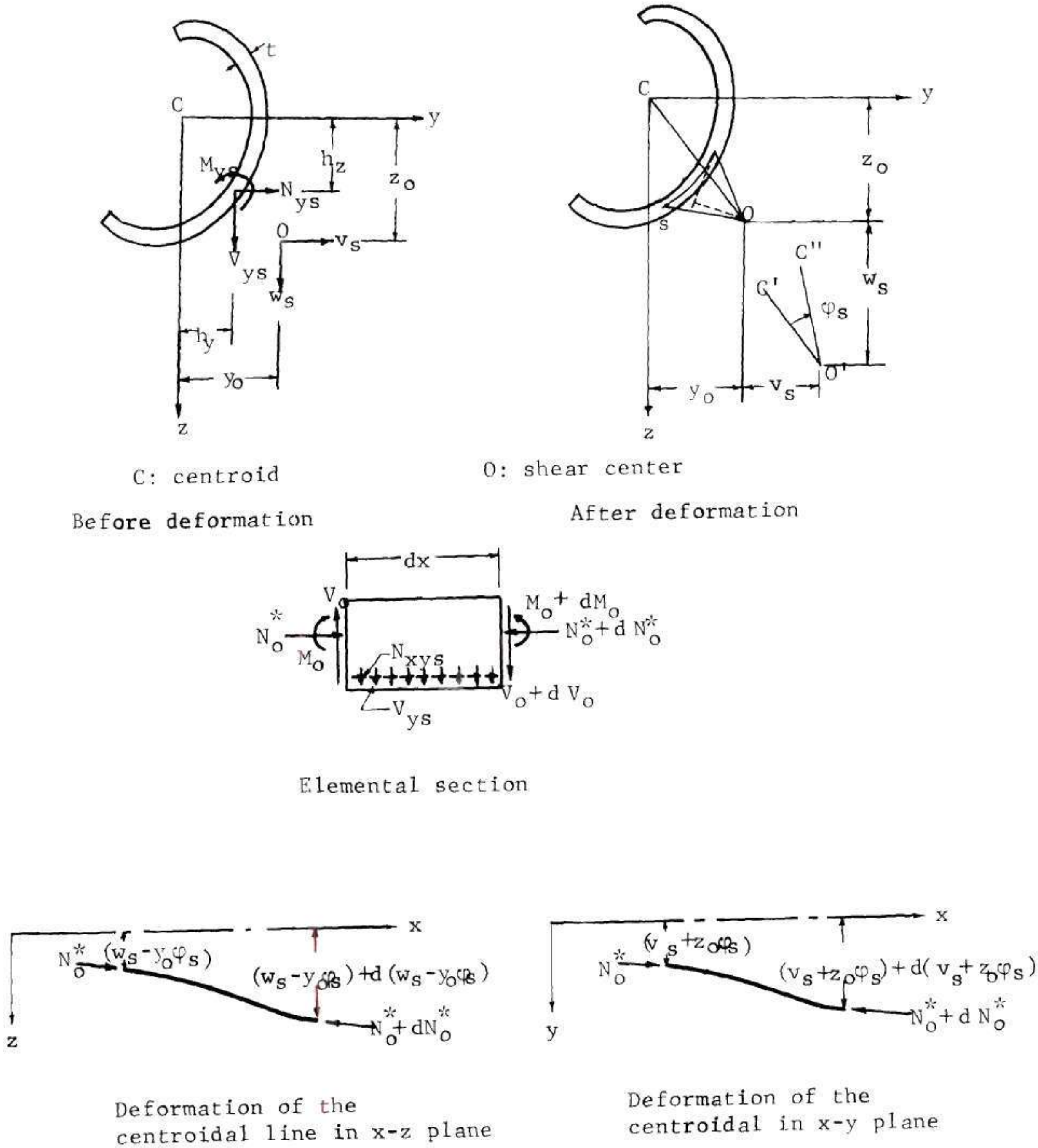


Figure 10. Stringer Deformation, Geometry and Coordinate

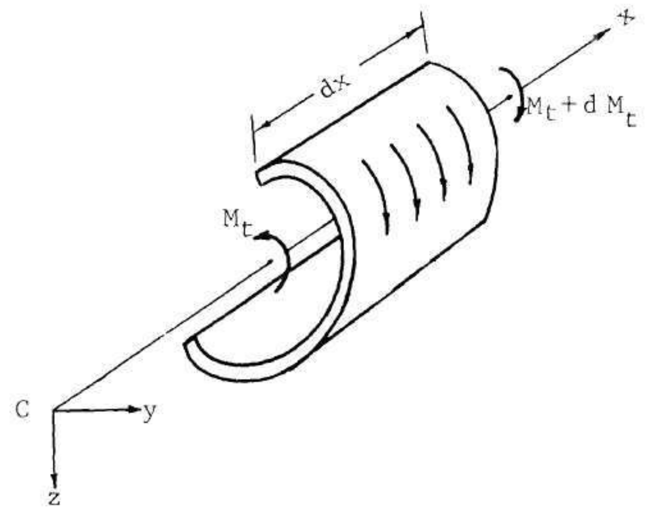


Figure 11. Torsion of a Stringer

APPENDIX C

ALTERNATIVE SOLUTION FOR DISPLACEMENT COMPONENTS u AND v

Consider the ordinary differential equation

$$\left(\frac{d^2}{dx^2} - a^2\right) y = R(x) \ . \qquad \text{(C-1)}$$

By the method of coefficient comparison, the particular solution, y_p, of this equation can be expressed as:

(i) if $R(x) \equiv A \cos \beta x$, then $$y_p = \frac{A \cos \beta x}{(\beta^2 + a^2)^2} \ ; \qquad \text{(C-2)}$$

if $R(x) = A \sin \beta x$, then $$y_p = \frac{A \sin \beta x}{(\beta^2 + a^2)^2} \ ; \qquad \text{(C-3)}$$

(ii) if $R(x) = A\, e^{\pm \alpha x}$, then $$y_p = \frac{A\, e^{\pm \alpha x}}{(\beta^2 - a^2)^2} \ ; \qquad \text{(C-4)}$$

(iii) if $R(x) = e^{\pm \gamma x}(A \cos \delta x + B \sin \delta x)$, then

$$y_p = e^{\pm \gamma x}\,(a^{\pm} \cos \delta x + b^{\pm} \sin \delta x) \ , \qquad \text{(C-5)}$$

where

$$a^{\pm} = \frac{Af_1 \mp Bf_2}{f_1^2 + f_2^2} \quad ; \qquad b^{\pm} = \frac{Bf_1 \pm Af_2}{f_1^2 + f_2^2} \quad ;$$

$$f_1 = (\gamma^4 - 6\gamma^2\delta^2 + \delta^4) - 2a^2(\gamma^2 - \delta^2) + a^4 \; ;$$

$$f_2 = 4\gamma\delta(\gamma^2 - \delta^2) - 2a^2(2\gamma\delta) \; ,$$

if $\quad R(x) = e^{\pm\gamma x}(A \sin\delta x + B\cos\delta x)$, then

$$y_p = e^{\pm\gamma x}(a^{\pm}\sin\delta x + b^{\pm}\cos\delta x) \tag{C-6}$$

where

$$a^{\pm} = \frac{Af_1 \pm Bf_2}{f_1^2 + f_2^2} \quad ; \qquad b^{\pm} = \frac{Bf_1 \mp Af_2}{f_1^2 + f_2^2} \; .$$

Let the form of the solution of the displacement component, u, be

$$\bar{u} = U_{m(\eta)} \cos\frac{m\pi\xi}{L} \; . \tag{C-7}$$

Substitution of Eqs.(C-7) and (3-17) into Eq.(2-38) results in

$$\left[\frac{d^2}{d\eta^2} - \left(\frac{m\pi}{L}\right)^2\right]^2 U_{m(\eta)} = \sum_{j=1}^{R_{1m}} -\left[\nu\left(\frac{m\pi}{L}\right)^3 + \frac{m\pi}{L}\alpha_{mj}^2\right]\left[A_j e^{-\alpha_{mj}\eta} + A_{j+R_{1m}} e^{\alpha_{mj}\eta}\right] \tag{C-8}$$

$$+ \sum_{j=1}^{R_{2m}} \left[\frac{m\pi}{L}\beta_{mj}^2 - \nu\left(\frac{m\pi}{L}\right)^3\right]\left[A_{j+a_m}\cos\beta_{mj} + A_{j+b_m}\sin\beta_{mj}\right]$$

$$+ \sum_{j=1}^{R_{3m}} \left[A_{j+c_m} e^{-\gamma_{mj}\eta} \left(d_{1mj} \cos \delta_{mj}\eta + d_{2mj} \sin \delta_{mj}\eta \right) \right.$$

$$+ A_{j+d_m} e^{-\gamma_{mj}\eta} \left(d_{1mj} \sin \delta_{mj}\eta - d_{2mj} \cos \delta_{mj}\eta \right)$$

$$+ A_{j+e_m} e^{\gamma_{mj}\eta} \left(d_{1mj} \cos \delta_{mj}\eta - d_{2mj} \sin \delta_{mj}\eta \right)$$

$$\left. + A_{j+f_m} e^{\gamma_{mj}\eta} \left(d_{1mj} \sin \delta_{mj}\eta + d_{2mj} \cos \delta_{mj}\eta \right) \right] ,$$

where

$$d_{1mj} = -\left[\nu \left(\frac{m\pi}{L}\right)^3 + \frac{m\pi}{L} \left(\gamma_{mj}^2 - \delta_{mj}^2 \right) \right] ; \qquad d_{2mj} = -2\, \frac{m\pi}{L} \gamma_{mj} \delta_{mj} .$$

By applying the previous results, the particular solution for Eq.(C-8) is

$$U_m(\eta) = \sum_{j=1}^{R_{1m}} \left[A_j P_{mj}^{(r)} e^{-\alpha_{mj}\eta} + A_{j+R_{1m}} P_{mj}^{(r)} e^{\alpha_{mj}\eta} \right] \tag{C-9}$$

$$+ \sum_{j=1}^{R_{2m}} \left[A_{j+a_m} P_{mj}^{(im)} \cos \beta_{mj}\eta + A_{j+b_m} \sin \beta_{mj}\eta \right]$$

$$+ \sum_{j=1}^{R_{3m}} \left[A_{j+c_m} e^{-\gamma_{mj}\eta} \left(P_{mj}^{(c)} \cos \delta_{mj}\eta + \bar{P}_{mj}^{(c)} \sin \delta_{mj}\eta \right) \right.$$

$$+ A_{j+d_m} e^{-\delta_{mj}\eta} \left(P_{mj}^{(c)} \sin \delta_{mj}\eta - \bar{P}_{mj}^{(c)} \cos \delta_{mj}\eta \right)$$

$$+ A_{j+e_m} e^{\gamma_{mj}\eta} \left(P_{mj}^{(c)} \cos \delta_{mj}\eta - \bar{P}_{mj}^{(c)} \sin \delta_{mj}\eta \right)$$

$$\left. + A_{j+f_m} e^{\gamma_{mj}\eta} \left(P_{mj}^{(c)} \sin \delta_{mj}\eta + \bar{P}_{mj}^{(c)} \cos \delta_{mj}\eta \right) \right] ,$$

where

$$p^{(r)}_{mj} = \frac{-\left[\nu\left(\frac{m\pi}{L}\right)^3 + \frac{m\pi}{L}\alpha^2_{mj}\right]}{\left[\alpha^2_{mj} - \left(\frac{m\pi}{L}\right)^2\right]^2} \; ; \qquad p^{(im)}_{mj} = \frac{\frac{m\pi}{L}\beta^2_{mj} - \nu\left(\frac{m\pi}{L}\right)^3}{\left(\beta^2_{mj} + \left(\frac{m\pi}{L}\right)^2\right)^2} \; ;$$

$$p^{(c)}_{mj} = \frac{d_{1mj} f_{1mj} + d_{2mj} f_{2mj}}{f^2_{1mj} + f^2_{2mj}} \; ; \qquad \bar{p}^{(c)}_{mj} = \frac{d_{2mj} f_{1mj} - d_{1mj} f_{2mj}}{f^2_{1mj} + f^2_{2mj}} \; ;$$

$$f_{1mj} = \left(\gamma^4_{mj} - 6\gamma^2_{mj}\delta^2_{mj} + \delta^4_{mj}\right) - 2\left(\frac{m\pi}{L}\right)^2\left(\gamma^2_{mj} - \delta^2_{mj}\right) + \left(\frac{m\pi}{L}\right)^4 \; ;$$

$$f_{2mj} = 4\gamma_{mj}\delta_{mj}\left(\gamma^2_{mj} - \delta^2_{mj}\right) - 4\left(\frac{m\pi}{L}\right)^2\gamma_{mj}\delta_{mj} \, .$$

Similarly, by letting the form of the solution of the displacement component, v, be

$$\bar{V} = V_m(\eta) \sin\frac{m\pi\xi}{L} \, , \tag{C-10}$$

one obtains

$$V_m(\eta) = \sum_{j=1}^{R_{1m}} \left[-A_j q^{(r)}_{mj} e^{-\alpha_{mj}\eta} + A_{j+R_{1m}} q^{(r)}_{mj} e^{\alpha_{mj}\eta}\right] \tag{C-11}$$

$$+ \sum_{j=1}^{R_{2m}} \left[-A_{j+a_m} q^{(im)}_{mj} \sin\beta_{mj}\eta + A_{j+b_m} q^{(im)}_{mj} \cos\beta_{mj}\eta\right]$$

$$+ \sum \left[A_{j+c_m} e^{-\gamma_{mj}\eta}\left(-q^{(c)}_{mj}\cos\delta_{mj}\eta - \bar{q}^{(c)}_{mj}\sin\delta_{mj}\eta\right)\right.$$

$$\left. + A_{j+d_m} e^{-\gamma_{mj}\eta}\left(-q^{(c)}_{mj}\sin\delta_{mj}\eta + \bar{q}^{(c)}_{mj}\cos\delta_{mj}\eta\right)\right.$$

$$+ A_{j+e_m} e^{\gamma_{mj}\eta} (q^{(c)}_{mj} \cos \delta_{mj}\eta - \bar{q}^{(c)}_{mj} \sin \delta_{mj}\eta)$$

$$+ A_{j+f_m} e^{\gamma_{mj}\eta} (q^{(c)}_{mj} \sin \delta_{mj}\eta + \bar{q}^{(c)}_{mj} \cos \delta_{mj}\eta)] ,$$

where

$$q^{(r)}_{mj} = \frac{[\alpha^3_{mj} - (2+\nu)(\frac{m\pi}{L})^2\alpha_{mj}]}{[\alpha^2_{mj} - (\frac{m\pi}{L})^2]^2} ; \qquad q^{(im)}_{mj} = \frac{-[\beta^3_{mj} + (2+\nu)(\frac{m\pi}{L})^2\beta_{mj}]}{[\beta^2_{mj} + (\frac{m\pi}{L})^2]^2} ;$$

$$q^{(c)}_{mj} = \frac{e_{1mj} f_{1mj} + e_{2mj} f_{2mj}}{f^2_{1mj} + f^2_{2mj}} ; \qquad \bar{q}^{(c)}_{mj} = \frac{e_{2mj} f_{1mj} - e_{1mj} f_{2mj}}{f^2_{1mj} + f^2_{2mj}} ;$$

in which

$$e_{1mj} = (\gamma^3_{mj} - 3\gamma_{mj}\delta^2_{mj}) - (2+\nu)(\frac{m\pi}{L})^2\gamma_{mj} ,$$

$$e_{2mj} = (3\gamma^2_{mj}\delta_{mj} - \delta^3_{mj}) - (2+\nu)(\frac{m\pi}{L})^2\delta_{mj} .$$

APPENDIX D

SOME DERIVATIVES OF FUNCTIONS $e^{\pm\alpha\xi} \cos \beta\xi$ AND $e^{\pm\alpha\xi} \sin \beta\xi$

$$\frac{d}{d\xi}\, e^{\pm\alpha\xi} \cos \beta\xi = e^{\pm\alpha\xi} \left[\pm\, \alpha \cos \beta\xi - \beta \sin \beta\xi\right] \;;$$

$$\frac{d^2}{d\xi^2}\, e^{\pm\alpha\xi} \cos \beta\xi = e^{\pm\alpha\xi} \left[(\alpha^2-\beta^2) \cos \beta\xi \mp 2\alpha\xi \sin \beta\xi\right] \;;$$

$$\frac{d^3}{d\xi^3}\, e^{\pm\alpha\xi} \cos \beta\xi = e^{\pm\alpha\xi} \left[\pm(\alpha^3-3\alpha\beta^2)\cos \beta\xi - (3\alpha^2\beta-\beta^3)\sin\beta\xi\right] \;;$$

$$\frac{d^4}{d\xi^4}\, e^{\pm\alpha\xi} \cos \beta\xi = e^{\pm\alpha\xi} \left[(\alpha^4-6\alpha^2\beta^2+\beta^4)\cos\beta\xi \mp (4\alpha^3\beta-4\alpha\xi^3)\sin\beta\xi\right] \;;$$

$$\frac{d^5}{d\xi^5}\, e^{\pm\alpha\xi} \cos \beta\xi = e^{\pm\alpha\xi}$$

$$\times \left[\pm(\alpha^5-10\alpha^3\beta^2+5\alpha\beta^4)\cos\beta\xi - (5\alpha^4\beta-10\alpha^2\beta^3+\beta^5)\sin\beta\xi\right] \;;$$

$$\frac{d^6}{d\xi^6}\, e^{\pm\alpha\xi} \cos \beta\xi = e^{\pm\alpha\xi}$$

$$\times \left[(\alpha^6-15\alpha^4\beta^2+15\alpha^2\beta^4-\beta^6)\cos\beta\xi \mp (6\alpha^5\beta-20\alpha^3\beta^3+6\alpha\beta^5)\sin\beta\xi\right] \;;$$

$$\frac{d^7}{d\xi^7} e^{\pm\alpha\xi} \cos \beta\xi = e^{\pm\alpha\xi}$$

$$\times [\pm(\alpha^7-21\alpha^5\beta^2+35\alpha^3\beta^4-7\alpha\beta^6)\cos\beta\xi - (7\alpha^6\beta-35\alpha^4\beta^3+21\alpha^2\beta^5-\beta^7)\sin\beta\xi] \; ;$$

$$\frac{d}{d\xi} e^{\pm\alpha\xi} \cos \beta\xi = e^{\pm\alpha\xi} [\pm\alpha \sin\beta\xi + \beta\cos\beta\xi] \; ;$$

$$\frac{d^2}{d\xi^2} e^{\pm\alpha\xi} \sin \beta\xi = e^{\pm\alpha\xi}[(\alpha^2-\beta^2)\sin\beta\xi \pm 2\alpha\beta\cos\beta\xi] \; ;$$

$$\frac{d^3}{d\xi^3} e^{\pm\alpha\xi} \sin \beta\xi = e^{\pm\alpha\xi}[\pm(\alpha^3-3\alpha\beta^2)\sin\beta\xi + (3\alpha^2\beta-\beta^3)\cos\beta\xi] \; ;$$

$$\frac{d^4}{d\xi^4} e^{\pm\alpha\xi} \sin \beta\xi = e^{\pm\alpha\xi}[\alpha^4-6\alpha^2\beta^2+\beta^4)\sin\beta\xi \pm (4\alpha^3\beta-4\alpha\beta^3)\cos\beta\xi] \; ;$$

$$\frac{d^5}{d\xi^5} e^{\pm\alpha\xi} \sin \beta\xi = e^{\pm\alpha\xi}$$

$$\times [\pm(\alpha^5-10\alpha^3\beta^2+5\alpha\beta^4)\sin\beta\xi + (5\alpha^4\beta-10\alpha^2\beta^3+\beta^5)\cos\beta\xi] \; ;$$

$$\frac{d^6}{d\xi^6} e^{\pm\alpha\xi} \sin \beta\xi = e^{\pm\alpha\xi}$$

$$\times [(\alpha^6-15\alpha^4\beta^2+15\alpha^2\beta^4-\beta^6)\sin\beta\xi \pm (6\alpha^5\beta-20\alpha^3\beta^3+6\alpha\beta^5)\cos\beta\xi] \; ;$$

$$\frac{d^7}{d\xi^7} e^{\pm\alpha\xi} \sin \beta\xi = e^{\pm\alpha\xi}$$

$$\times \left[\pm(\alpha^7-21\alpha^5\beta^2+35\alpha^3\beta^4-7\alpha\beta^6)\sin\beta\xi + (7\alpha^6\beta-35\alpha^4\beta^3+21\alpha^2\beta^5-\beta^7)\cos\beta\xi\right] ;$$

APPENDIX E

COMPUTER PROGRAM

A computer program to calculate the critical load of a longitudinally stiffened cylindrical shell subjected to an axial compressive force is presented in this appendix. The program, which is written in ALGOL for use with UNIVAC 1108 digital computer and applicable only to the cylindrical shell stiffened by equally spaced stringers of same size, involves both the linear analysis and the analysis including the pre-buckling deformation effect. If "OPT" is assigned to be 1 in the program, only the linear analysis is considered, otherwise (OPT=2) the analysis including the pre-buckling deformation effect will be considered.

In the program, the procedures (subroutines) used for calculating a determinant value (named DETERMINT) and for solving simultaneous linear equations (named SOLVE), respectively, are the existing ones originally supplied by Burroughs Company. All the procedures and real procedures (functions) used in the program are listed in Table 5. The flow chart of the main program is presented in Figure 12. Table 6 shows a comparison between main notations used in the program and that used in the text.

Table 5. List of Program Procedures

Name	Description
SOLVE	This procedure solves a set of N by N simultaneous linear equations using the Crout reduction and is written by R. D. Rodman, professional services divisional group of Burroughs Company.
DETERMINT	This procedure computes the determinant of a matrix of Nth order and is written by R. D. Rodman, professional services divisional group of Burroughs Company.
F	This real procedure calculates the post-buckling deformation and its derivatives.
GG	This real procedure calculates the function $\cos(2n\pi(y-d/2)/d)$ and its derivatives.
SIMP	This real procedure makes an numerical integral by using Simpson's Rule.
INSTIN	This procedure calculates the integral of Eq. (4-46)

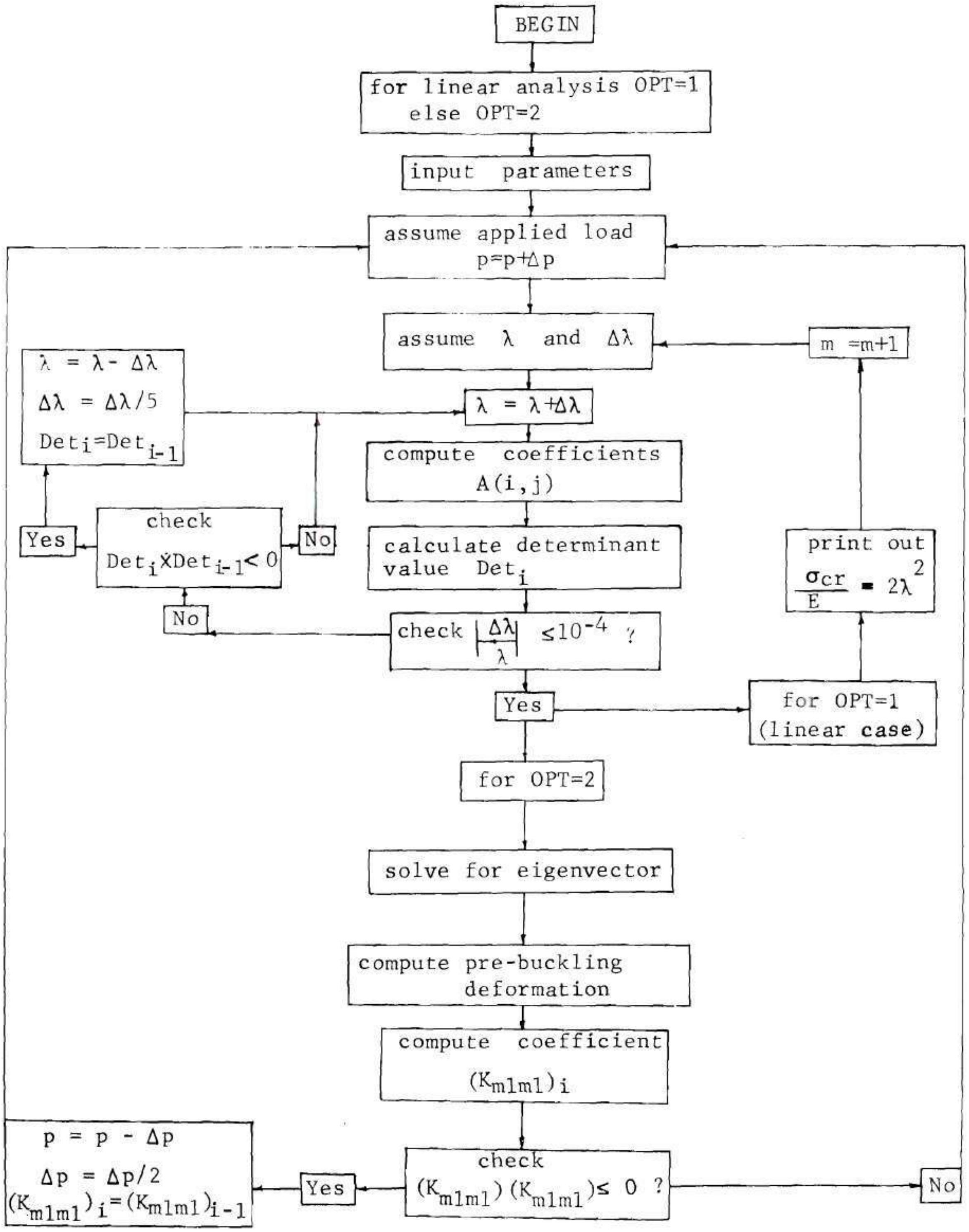


Figure 12. Flow Chart of Main Program

Table 6. Comparison Between Text and Program Notation

Program	Text	Program	Text
LL	1/a	HR	a/h
ARH	A_s/ah	IYI	I_y/ah^3
IZI	I_z/a^3h	IJI	J/ah^3
C1B	$\bar{C}_1$	CB	C
ZI, HZ	$\bar{h}_z=h_z/a$	YO	$\bar{y}_o=y_o/a$
HY	$\bar{h}_y=h_y/a$	ZO	$\bar{z}_o=z_o/a$
PE	$\sigma=(\frac{1}{2}\,\frac{P}{E})^{\frac{1}{2}}$	DP	$\Delta\sigma$
LA	λ (=σ linear analysis)	DLA	$\Delta\lambda$
EI	E_s/E	NU	
BI	total number of stringers	BS	total number of terms in Simpson's Rule
BN	upper limit of m in postbuckling def.	BM	upper limit of n in postbuckling def. must 1 in OPT 1
BIGN	upper limit of m' in Eq. (4-48)	BIGM	upper limit of n' in Eq. (4-48)

```
BEGIN
COMMENT THIS PROGRAM CALCULATES THEN BUCKLING LOAD OF THE LONGITUDINALLY
   STIFFENED CYLINDRICAL SHELLS,  IF OPT=1 LINEAR ANALYSIS ONLY,
   IF  OPT=2 IT INCLUDES THEN EFFECT OF PRE-BUCKLING DEFORMATION--
   TRIAL AND ERROR METHOD IS USED ;
REAL RJ,RK,CHB,CHJ,SHJ,SHB,ZOS,ZOP,ZOM,DEN ;
REAL LPI,DKL,WN,MPI,DNM,MPP,DNP,YY ;
REAL  HR,EI,GI,IYI,IZI,ARH,GYB,GZB,D2, DBN,D3,  ZI,DDN,ZO,ISN,ICS ;
REAL PE,DP,PE2,PI,LL,LA,LA1,DLA,NPT,F1,NU,DAN,DY,W,CSJ,SNJ,CSB,SNB,JCN,
     R1,D1,IYB,IZB,R2,R3,A1B,A2B,HY,HZ,GY,GZ,HYB,HZB,YO,ZO,C1B,CB,IOS,
     KPI,NPP,YI,SUM1,SUM2,SUM3,SUM4,SUM5,YT,H,SUM6,SUM,IJI,CNM ;
INTEGER KK,NC,KM,NC1,BI,BN,BM,ND,N,M,NM,K1,J,I,ROW,COL,S,JI,KL,NP,MP,
     NMP,BS,LQ,K,L,COD,IT,KO,JCKL,JCKM,JCNM,JCNP,COJ,J1,J2,K2,NP2,IN,
     BIGN,BIGM,NE,OPT,N2 ;
FORMAT FM1(X5,"STRINGER:TOTAL NO=",I2,X3,"NO OF REPEATED=",I2,A1) ,
       FM2(X3,"L/R =",D7.3,X5,"H/R=",D8.5,A1.1) ,
       FM3(X14,"ZI/R=",D7.4,X5,"AI/HxR=",D7.4,X5,"IY/RxH*3=",D7.4,X5,
            "IZ/HxR*3=",D7.4,X5,"JT/RxH*3=",D7.4,A1),
       FM4(X14,"EI/E =",D7.3,X5,"GI/E =",D7.3,A1) ,
       FM5(X8,"LOAD=",R12.5,X3,"INCRE=",R12.5,X3,"DETER=",R12.5,A1.2) ,
       FT5(X8,"LA=",R12.5,X3,"DLA=",R12.5,X5,"DETER=",R12.5,A1),
       FM6(X5,"NO OF TERM IN X=",I2,X5,"NO OF TERM IN Y=",I2,A1),
       FM7(X5,"N",X4,"M",X4,"I",X7,"A",X11,"B",X11,"C",X11,"D",X11,"E",
          X11,"F",X11,"G",X11,"H",A1) ,
       FM8 (3(X3,I2),8(X2,R12.5),A1) ,
       FM11(X10,"COEF MATRIX IS SINGULAR",A1) ,
       FM12(X10,"NOT CONVERGENT",A1) ,
       FM15(X5,"N =",I2,X5,"EIGENVALUE =",R12.5,A2.2) ,
       FM16(E1,"LINEAR STABILITY ANALYSIS OF SIMPLY SUPPORTED CASE",
            A1.1) ,
       FM17(E1,"STABILITY ANALYSIS WITH PRE-BUCKLING DEFORMATION EFFECT"
            A1.1) ,
       FM9(X5,"N=",I2,X3,"M=",I2,X3,"AB=",R12.5,A1) ;


PROCEDURE SOLVE(N, A, C,RSN,E,K1,EPS, X,E1,E2) ;
COMMENT  THIS PROCEDURE SOLVES A SET OF N BY N SIMULAEOUS LINEAR
```

```
        EQUATIONS USING THE CROUT REDUCTION. PROVISION IS MADE FOR
        EASILY SOLVING SEVERAL SETS OF SUCH EQUATIONS, THE COEFFICIENT
        MATRIX OF WHICH REMAINS THE SAME. PROVISION IS ALSO MADE FOR
        ITERATING THE SOLUTION TO REDUCE ROUND-OFF ERROR ;
  VALUE N,RSW,E,K1,EPS ; INTEGER N,K1 ; REAL E,EPS ; BOOLEAN RSW ;
  REAL ARRAY  A,C,X ;   LABEL E1,E2 ; BEGIN
   INTEGER I,J,K,K2,L,J1 ; REAL 2 BIG,TEMP,DIAG,NORM,Q ;
   OWN INTEGER ARRAY F[0:N] ;     REAL ARRAY D[0:N],B[0:N,0:N] ;
S1: IF RSW THEN GO TO REP ;
COMMENT THE COEFFICIENT MATRIX IS TRIANGULARIZED. ;
    FOR I=(1,1,N) DO FOR J=(1,1,N) DO B[I,J]=A[I,J] ;
S2: FOR I=(1,1,N) DO BEGIN L=I-1 ;
    FOR J=(I,1,N) DO BEGIN Q=0 ; FOR K=(1,1,L) DO Q=B[J,K]*B[K,I]+Q ;
    B[J,I]=B[J,I]-Q  END ; BIG=0 ; K2=I ;
S3: FOR K=(I,1,N) DO BEGIN IF ABS(B[K,I]) GTR BIG THEN BEGIN
    BIG=ABS(B[K,I]) ; K2=K END END ;
COMMENT E1 IS THE NON-LOCAL LABEL T WHICH AN EXIT IS MADE IF THE
       COEFFICIENT MATRIX IS SINGULAR ;
S4: IF BIG LEQ EPS THEN GO TO E1 ; F[I]=K2 ;
    IF K2 NEQ I THEN
S5: FOR K=(1,1,N) DO BEGIN TEMP=A[K2,K] ;
    A[K2,K]=A[I,K] ; A[I,K]=TEMP ; TEMP=B[K2,K] ; B[K2,K]=B[I,K] ;
    B[I,K]=TEMP END ; DIAG=B[I,I] ;
S6: FOR J=(I+1,1,N) DO BEGIN Q=0 ; FOR K=(1,1,L) DO Q=B[I,K]*B[K,J]+Q ;
    B[I,J]=(B[I,J]-Q)/DIAG END END ;
REP: FOR I=(1,1,N) DO BEGIN TEMP=C[F[I]] ; C[F[I]]=C[I] ;
    D[I]=C[I]=TEMP END ;
    FOR I=(1,1,N) DO BEGIN L=I-1 ; Q=0 ;
S7: FOR K=(1,1,L) DO Q=B[I,K]*D[K]+Q ; D[I]=(D[I]-Q)/B[I,I] END ;
S8: FOR I=(N,-1,1) DO BEGIN Q=0 ; FOR K=(I+1,1,N) DO Q=B[I,K]*X[K]+Q ;
    X[I]=D[I]-Q END ;
S9: IF E EQL 0 THEN GO TO EXIT ; J1=0 ;
COMMENT THE SOLUTION IS ITERATED AND TESTED FOR ACCURACY ;
IT1: IF J1 GTR K1 THEN GO TO E2 ; NORM=0 ;
    FOR I=(1,1,N) DO BEGIN Q=0 ; L=I-1 ;
S10: FOR K=(1,1,N) DO Q=A[I,K]*X[K]+Q ; D[I]=C[I]-Q ;
S11: NORM=ABS(D[I])+NORM ; Q=0 ;
```

```
S12: FOR K=(1,1,L) DO Q=B[I,K]*D[K]+Q ; D[I]=(D[I]-Q)/B[I,I] END ;
    FOR I=(N,-1,1) DO BEGIN Q=0 ;
S13: FOR K=(I+1,1,N) DO Q=B[I,K]*D[K]+Q ; X[I]=X[I]+D[I]-Q END ;
S14: J1=J1+1 ;
S15: IF N*E LSS NORM THEN GO TO IT1 ;
EXIT: END OF SOLVE ;


PROCEDURE DETERMINT(N,AA,EPS,DD) ;
  COMMENT  THIS PROCEDURE CALCULATES THE DETERMINANT VALUE ;
  VALUE N,EPS ; INTEGER N ; REAL EPS,DD  ; REAL ARRAY AA ;
   BEGIN INTEGER I,J,K,K2,L ; REAL 2 BIG,DIAG,TEMP,Q,D ;
   REAL   ARRAY A[0:N,0:N] ;
   D=1 ; FOR I=(1,1,N) DO FOR J=(1,1,N) DO A[I,J]=AA[I,J] ;
D1: FOR I=(1,1,N) DO BEGIN L=I-1 ;
    FOR J=(I,1,N) DO BEGIN Q=0 ; FOR K=(1,1,L) DO Q=A[J,K]*A[K,I]+Q ;
    A[J,I]=A[J,I]-Q END ;
    BIG=0 ; K2=I ;
D2: FOR K=(I,1,N) DO BEGIN
    IF ABS(A[K,I]) GTR BIG THEN BEGIN BIG=ABS(A[K,I]) ; K2=K END END ;
    IF BIG LSS EPS THEN BEGIN D=0 ; GO TO EXIT END ;
    IF K2 NEQ I THEN BEGIN D=-D ; FOR K=(1,1,N) DO BEGIN
    TEMP=A[I,K] ; A[I,K]=A[K2,K] ; A[K2,K]=TEMP END END ;
D3: DIAG=A[I,I] ; D=D*DIAG ;
    FOR J=(I+1,1,N) DO BEGIN Q=0 ; FOR K=(1,1,L) DO Q=A[I,K]*A[K,J]+Q ;
    A[I,J]=(A[I,J]-Q)/DIAG END END ;   DD=D ;
EXIT: END OF DETERM ;


REAL PROCEDURE F(I,J1,R,A,YY) ;
  VALUE I,J1,YY ;      INTEGER I,J1;  REAL YY ; ARRAY A,R ;
  BEGIN  INTEGER S,S1 ; REAL  SAJ,CHJ,DAN,DBN,SNJ,CSJ,SUM ;
  SUM=0 ;
    IF J1 EQL 0 THEN BEGIN FOR S=1,2 DO BEGIN S1=S+2 ;
   CHJ=EXP(R[S]*(YT-DY)) ; SHJ=EXP(-R[S]*YT) ;
   CSJ=COS(R[S1]*YY) ; SNJ=SIN(R[S1]*YY) ;
  IF I EQL 0 THEN  SUM=SUM+SHJ*(A[S]*CSJ+A[S+2]*SNJ)+CHJ*(A[S+4]*CSJ
```

```
+A[S1+4]*SNJ) ELSE BEGIN IF I EQL 1 THEN SUM=SUM+SHJ*(-A[S]*(R[S]*CSJ
 +R[S1]*SNJ)+A[S+2]*(-R[S]*SNJ+R[S1]*CSJ))+CHJ*(A[S+4]*(R[S]*CSJ-R[S1]
 *SNJ)+A[S+6]*(R[S]*SNJ+R[S1]*CSJ)) ELSE BEGIN
 IF I EQL 2 THEN BEGIN DAN=R[S]**2-R[S1]**2 ;
DBN=2*R[S]*R[S1] END ; IF I EQL 4 THEN BEGIN DAN=R[S]**4-6*(R[S]*R[S1]
)**2+R[S1]**4 ; DBN=4*R[S]*R[S1]*(R[S]**2-R[S1]**2) END ;
SUM=SUM+SHJ*(A[S]*(DAN*CSJ+DBN*SNJ)+A[S+2]*(DAN*SNJ-DBN*CSJ))+CHJ*(
A[S+4]*(DAN*CSJ-DBN*SNJ)+A[S+6]*(DAN*SNJ+DBN*CSJ)) END ; END ;
 END END ELSE BEGIN
 FOR S=(1,1,J1) DO BEGIN SHJ=EXP(-R[S]*DY) ; CHJ=EXP(R[S]*(YT-DY)) ;
 SUM=SUM+A[S]*(-R[S])**I*SHJ+A[S1]*CHJ*R[S]**I END ;
 FOR S=(J1+1,1,4) DO BEGIN S1=S+4-J1 ; SNJ=SIN(R[S]*YT) ;
 CSJ=COS(R[S]*YT) ; IF I EQL 0 OR I EQL 4 OR I EQL 6 THEN
 SUM=SUM+(A[S]*CSJ+A[S1]*SNJ)*(R[S]**I) ;
IF I EQL 2 OR I EQL 6 THEN SUM=SUM-(A[S]*CSJ+A[S1]*SNJ)*(R[S]**I) END;
  END ; F=SUM ;
END OF F ;


REAL PROCEDURE GG(N,M,DY,YT) ;
  VALUE N,M,DY,YT ; INTEGER N,M ; REAL DY,YT ;
  BEGIN REAL PI,DAN,DBN ; PI=3.1415927 ; DAN=2*M*PI/DY ;
   IF M NEQ 0 THEN BEGIN
   IF N EQL 0 THEN DBN=COS(DAN*(YT-DY/2) ) ;
   IF N EQL 2 OR N EQL 6 THEN DBN=-(DAN**N)*COS(DAN*(YT-DY/2)) ;
   IF N EQL 4 OR N EQL 8 THEN DBN=(DAN**N)*COS(DAN*(YT-DY/2));
   IF N EQL 1 OR N EQL 5 THEN DBN=-(DAN**N)*SIN(DAN*(YT-DY/2)) ;
   IF N EQL 3 OR N EQL 7 THEN DBN=(DAN**N)*SIN(DAN*(YT-DY/2)) ;
   END ELSE BEGIN IF N EQL 0 THEN DBN=1 ELSE DBN=0 END ;
   GG=DBN END OF GG ;


REAL PROCEDURE SIMP(N,H,Y) ;
  VALUE N,H ;    INTEGER N ; REAL H ; REAL ARRAY Y ;
  BEGIN  REAL SUM ; INTEGER S ;  SUM=Y[0]+Y[N] ;
   FOR S=(1,2,N-1) DO SUM=SUM+4*Y[S] ;
   FOR S=(2,2,N-2) DO SUM=SUM+2*Y[S] ;
```

```
   SIMP=SUM*H/3 ;
 END OF SIMP ;


PROCEDURE  INTSIN(N,M,K,L,ISN,ICS) ;
  VALUE N,M,K,L ;  INTEGER N,M,K ;  REAL L,ISN,ICS ;
  BEGIN  REAL IS1,IS2,IS3,IS4,PI ; INTEGER NM,NM2 ; PI=3.1415927 ;
    NM=K+M-N ;  NM2=2*ENTIER(NM/2+0.55) ;
    IF NM EQL 0 THEN IS1=L/4 ELSE IF NM EQL NM2 THEN IS1=0 ELSE
    IS1=L/(2*NM*PI) ;
    NM=K-M+N ; NM2=2*ENTIER(NM/2+0.55) ;
    IF NM EQL 0 THEN IS2=L/4 ELSE IF NM EQL NM2 THEN IS2=0 ELSE
    IS2=L/(2*NM*PI) ;
    NM=K+M+N ; NM2=2*ENTIER(NM/2+0.55) ;
    IF NM EQL NM2 THEN IS3=0 ELSE IS3=L/(2*NM*PI) ;
    NM=M+N-K ; NM2=2*ENTIER(NM/2+0.55) ;
    IF NM EQL 0 THEN IS4=L/4 ELSE IF NM EQL NM2 THEN IS4=0 ELSE
    IS4=L/(2*NM*PI) ; ISN=IS1+IS2-IS3+IS4 ; ICS=IS1+IS2+IS3-IS4
  END OF INTSIN ;

COMMENT  GIVEN DATA ;
    OPI=1 ;
    OPI=2 ;
    BN=1 ; BM=1 ; BS=40 ;
    LL=3 ; HR=1/402 ; BI=0 ; EI=4.2678 ; NU=0.3 ;  GI=EI/(2*(1+NU)) ;
    ARH=0.942187 ; ZI=0.07574 ;
    IYI=17.844 ; IZI=0.000307 ;IJI=44.6655 ;
    C1B=0 ; NN=0 ;
    ZI=-ZI ;
    YO=ZO=HY=0 ; HZ=ZI ;
    R1=R2=0 ;
    BIGN=10; BIGM=11; NE=BIGN*BIGM ;
    TN=2 ;
    TN=3 ;
    TN=1 ;

COMMENT  BEGINING OF CALCULATION ;
```

```
     IF OPT EQL 1 THEN WRITE(FM16) ; IF OPT EQL 2 THEN WRITE(FM17) ;
     PI=3.1415927 ;
     R3=12/(HR**2) ; GY=Y0-HY ; GZ=Z0-HZ ;
     GYB=R3*GY ; GZB=R3*GZ ; HZB=HZ*R3 ; HYB=HY*R3 ;
     D1=GY-R1*GZ ; D2=R2*GY-GZ ;  D3=HZ*GY-HY*GZ+WN ;
     A1B=12*(1-NU*NU)*EI*ARH/(HR**2) ; A2B=(1-NU*NU)*EI*ARH ;
     CB=6*(1-NU)*IJT*EI ; I0S=12*(1-NU*NU)*IJT-A1B*(GY*Y0+GZ*Z0) ;
     IYB=12*(1-NU**2)*IYI*EI ; IZB=(1-NU**2)*IZI*EI ;
     WRITE(LL,HR,FM2) ; WRITE(BI,FM1) ;
     WRITE(ZI,ARH,IYI,IZI,IJI,FM3) ; WRITE(EI,GI,FM4) ;
     IF OPT EQL 2 THEN WRITE(TN,BM,FM6) ;
     DY=2*PI/BI ; NC=8*BI ; NC1=NC-1 ; ND=BN*BM ;
     W=0 ; H=DY/BS ;
     Z0=SQRT(12*(1-NU**2))/HR ;
BEGIN
REAL ARRAY AU,AV,NXY,NY,VY,MY[0:4],V[0:9],A[0:NC,0:NC],DTR,DFT[0:40],
            Y[0:BS],W[0:ND],AB[0:ND,0:8,0:BI],DA1,DA2,DA3,DA4[0:6] ;
INTEGER ARRAY JC[0:NC] ;
REAL ARRAY R,RKL,RNM,RNP,RKH[0:4],RR[0:ND,1:4],DC[0:NE]  ;
REAL ARRAY C,X[0:NC],Y1,Y2,Y3[0:BS],AC[0:NE,0:NE] ;
BOOLEAN RSW ;
LOCAL LABEL L3,   F1,F2,E1  ;
   DTR[1]=0 ;
   DTR[1]=-1 ;
   DTR[1]=1 ;
    KK=0 ;
    KK=1 ;

COMMENT  FIRST ASSUMED APPLIED LOAD ;
     PE=0.017 ; DP=0.001 ;

L1: KK=KK+1 ; PE=PE+DP ;  PE2=2*PE*PE ;
COMMENT START TO CALCULATE POS-BUCKLING DEFORMATION ;
     FOR N=(1,1,BN) DO BEGIN NPI=N*PI/LL ;
     IF OPT EQL 2 THEN NPI=TN*PI/LL ;

COMMENT  FIRST ASSUMED EIGENVALUE ;
```

```
    LA=0.017 ;
    LA=0.0205 ;

    FOR M=(1,1,BM) DO BEGIN NM=(N-1)*BM+M ;
E1: KI=0 ; DLA=0.001 ;
    DLA=0.0001 ;
L2: KI=KI+1 ; LA=LA+DLA ; ZOS=1-1/(Z0**2*LA**4) ; CHO=0 ;
    IF OPT EQL 1 THEN BEGIN PF=LA ; PE2=2*LA**2 END ;
    FOR K=(1,1,NC) DO FOR L=(1,1,NC) DO A[K,L]=0 ;
    J1=J2=0 ;
    IF ZOS LSS 0 THEN BEGIN  J1=J2=0 ; DAN=Z0*LA*NPI*SQRT((1+1/(Z0*LA*LA
    ))/2) ; ZOP=NPI*NPI+DAN ; ZOM=NPI*NPI-DAN ; DCN=(Z0*LA*NPI)**2*(1/(
    Z0*LA*LA)-1)/2 ; DBN=ZOP*ZOP+DCN ; R[1]=SQRT((SQRT(DBN)+ZOP)/2) ;
    R[3]=SQRT((SQRT(DBN)-ZOP)/2) ; DBN=ZOM*ZOM+DCN ;
    R[2]=SQRT((SQRT(DBN)+ZOM)/2) ; R[4]=SQRT((SQRT(DBN)-ZOM)/2) END ELSE
    BEGIN ZOP=Z0*LA*SQRT(1+SQRT(ZOS)) ; ZOM=Z0*LA*SQRT(1-SQRT(ZOS)) ;
    IF NPI LSS ZOM THEN BEGIN J1=J2=2 ; R[1]=SQRT(NPI*(NPI+ZOP)) ;
    R[2]=SQRT(NPI*(NPI+ZOM)) ; R[3]=SQRT(NPI*(ZOP-NPI)); R[4]=SQRT(NPI*(
    ZOM-NPI)) END ; IF NPI GTR ZOP THEN BEGIN J1=4 ; J2=0 ;
    R[1]=SQRT(NPI*(NPI+ZOP)) ; R[2]=SQRT(NPI*(NPI+ZOM)) ; R[3]=SQRT(NPI*
    (NPI-ZOP)) ; R[4]=SQRT(NPI*(NPI-ZOM)) END ; IF NPI GTR ZOM AND
    NPI LSS ZOP THEN BEGIN J1=3 ; J2=1 ; R[1]=SQRT(NPI*(NPI+ZOP)) ;
    R[2]=SQRT(NPI*(NPI+ZOM)) ; R[3]=SQRT(NPI*(NPI-ZOM)) ;
    R[4]=SQRT(NPI*(ZOP-NPI)) END END ;
    IF (J1+J2) EQL 0 THEN BEGIN
    FOR J=1,2 DO BEGIN K=J+2 ; RJ=R[J] ; RK=R[K] ;
    AU[J]=((RJ**6-15*RJ**4*RK**2+15*RJ**2*RK**4-RK**6)/(NPI**3)+(NU-2)*(
    RJ**4-6*(RJ*RK)**2+RK**4)/NPI+NPI*(1-2*NU)*(RJ**2-RK**2)+NU*NPI**3)/
    (Z0**2)-2*LA*LA*((RJ**2-RK**2)/NPI+NPI*NU) ;
    AU[K]=((6*RJ**5*RK-20*(RJ*RK)**3+6*RJ*RK**5)/(NPI**3)+(NU-2)* 4*(RJ
    **3*RK-RJ*RK**3)/NPI+NPI*(1-2*NU)*2*RJ*RK)/(Z0**2)-4*LA**2*RJ*RK/NPI
  ; AV[J]=-((RJ**7-21*RJ**5*RK**2+35*RJ**3*RK**4-7*RJ*RK**6)/(NPI**4)-(4
    +NU)*(RJ**5-10*RJ**3*RK**2+5*RJ*RK**4)/(NPI**2)+(5+2*NU)*(RJ**3-3*RJ
    *RK**2)-NPI**2*(2+NU)*RJ)/(Z0**2)+2*LA**2*((RJ**3-3*RJ*RK**2)/(NPI*
    NPI)-(2+NU)*RJ); AV[K]=-((7*RJ**6*RK-35*RJ**4*RK**3+21*RJ**2*RK**5-
    RK**7)/(NPI**4)-(4+NU)*(5*RJ**4*RK-10*RJ**2*RK**3+RK**5)/(NPI**2)+(5
    +2*NU)*(3*RJ**2*RK-RK**3)-NPI*NPI*RK*(2+NU))/(Z0*Z0)+2*LA*LA*((3*RJ
```

```
**2*RK-RK**3)/(NPI*NPI)-(2+NU)*RK) ; MY[J]=NU*NPI**2-(RJ**2-RK**2) ;
MY[K]=-2*RJ*RK ; VY[J]=(2-NU)*NPI*NPI*RJ-RJ**3+3*RJ*RK**2 ;
VY[K]=-(2-NU)*NPI*NPI*RK    +3*RJ**2*RK-RK**3 ; NY[J]=RJ*AV[J]-RK*
AV[K]-NU*NPI*AU[J]-1 ; NY[K]=RJ*AV[K]+RK*AV[J]-NU*NPI*AU[K] ;
NXY[J]=(1-NU)*(RJ*AU[J]-RK*AU[K]+NPI*AV[J])/2 ;
NXY[K]=(1-NU)*(RJ*AU[K]+RK*AU[J]+NPI*AV[K])/2 END ;
L6: FOR I=(1,1,BI) DO BEGIN ROW=(I-1)*8 ;
IF I EQL 1 THEN COL=(BI-1)*8 ELSE COL=(I-2)*8 ;
FOR J=1,2 DO BEGIN K=J+2 ; RJ=R[J] ; RK=R[K] ;
CSJ=COS(RK*(I-1)*DY) ; SNJ=SIN(RK*(I-1)*DY) ; CHB=SHJ=1 ;
CHJ=SHB=EXP(-RJ*DY) ; IF I GEQ 2 THEN BEGIN CSB=CSJ ; SNB=SNJ END
ELSE BEGIN CSB=COS(RK*BI*DY) ; SNB=SIN(RK*BI*DY) END ;
A[ROW+1,ROW+J]=SHJ*CSJ ; A[ROW+1,COL+J]=-SHB*CSB ;
A[ROW+1,ROW+2+J]=SHJ*SNJ ; A[ROW+1,COL+2+J]=-SHB*SNB ;
A[ROW+1,ROW+4+J]=CHJ*CSJ ; A[ROW+1,COL+4+J]=-CHB*CSB ;
A[ROW+1,ROW+6+J]=CHJ*SNJ ; A[ROW+1,COL+6+J]=-CHB*SNB ;
DAN=AU[J]*CSJ ; DBN=AU[K]*SNJ ; DCN=AU[J]*SNJ ; DDN=AU[K]*CSJ ;
A[ROW+2,ROW+J]=-SHJ*(DAN+DBN) ; A[ROW+2,ROW+4+J]=-CHJ*(DAN-DBN) ;
A[ROW+2,ROW+2+J]=-SHJ*(DCN-DDN) ; A[ROW+2,ROW+6+J]=-CHJ*(DDN+DCN) ;
DAN=AU[J]*CSB ; DBN=AU[K]*SNB ; DCN=AU[J]*SNB ; DDN=AU[K]*CSB ;
A[ROW+2,COL+J]=SHB*(DAN+DBN) ; A[ROW+2,COL+4+J]=CHB*(DAN-DBN) ;
A[ROW+2,COL+2+J]=SHB*(DCN-DDN) ; A[ROW+2,COL+6+J]=CHB*(DCN+DDN) ;
DAN=AV[J]*CSJ ; DBN=AV[K]*SNJ ; DCN=AV[J]*SNJ ; DDN=AV[K]*CSJ ;
A[ROW+3,ROW+J]=SHJ*(DAN+DBN) ; A[ROW+3,ROW+4+J]=-CHJ*(DAN-DBN) ;
A[ROW+3,ROW+2+J]=-SHJ*(DDN-DCN) ; A[ROW+3,ROW+6+J]=-CHJ*(DCN+DDN) ;
DAN=AV[J]*CSB ; DBN=AV[K]*SNB ; DCN=AV[J]*SNB ; DDN=AV[K]*CSB ;
A[ROW+3,COL+J]=-SHB*(DAN+DBN) ; A[ROW+3,COL+4+J]=CHB*(DAN-DBN) ;
A[ROW+3,COL+2+J]=SHB*(DDN-DCN) ; A[ROW+3,COL+6+J]=CHB*(DCN+DDN) ;
DAN=RJ*CSB ; DBN=RK*SNB ; DCN=RJ*SNB ; DDN=RK*CSB ;
A[ROW+4,COL+J]=-SHB*(DAN+DBN) ; A[ROW+4,COL+4+J]=CHB*(DAN-DBN) ,
A[ROW+4,COL+2+J]=SHB*(DDN-DCN) ; A[ROW+4,COL+6+J]=CHB*(DCN+DDN) ;
DAN=RJ*CSJ ; DBN=RK*SNJ ; DCN=RJ*SNJ ; DDN=RK*CSJ ;
A[ROW+4,ROW+J]=SHJ*(DAN+DBN) ; A[ROW+4,ROW+4+J]=-CHJ*(DAN-DBN) ;
A[ROW+4,ROW+2+J]=SHJ*(DCN-DDN) ; A[ROW+4,ROW+6+J]=-CHJ*(DCN+DDN) ;
DAN=(VY[J]-NPI*NXY[J]*HZB)*CSB ; DBN=(VY[K]+NPI*NXY[K]*HZB)*SNB ;
A[ROW+5,COL+J]=SHB*(DBN-DAN) ; A[ROW+5,COL+4+J]=CHB*(DAN+DBN) ;
DAN=(VY[J]-NPI*NXY[J]*HZB)*SNB ; DBN=-(VY[K]+NPI*NXY[K]*HZB)*CSB ;
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A[ROW+5,COL+2+J]=SHB*(DBN-DAN) ; A[ROW+5,COL+6+J]=CHB*(DAN+DBN) ;
DCN=IYB*NPI**4-2*A1B*(PE*NPI)**2 ;
DDN=IYB*NPI**4*(R1*AV[J]+D1*RJ)+2*A1B*(PE*NPI)**2*RJ*HY ;
DEN=IYB*NPI**4*(R1*AV[K]+D1*RK)+2*A1B*(PE*NPI)**2*RK*HY ;
DAN=VY[J]-HZB*NPI*NXY[J] ; DBN=VY[K]+HZB*NPI*NXY[K] ;
A[ROW+5,ROW+J]=SHJ*((DCN-DDN+DAN)*CSJ-(DEN+DBN)*SNJ) ;
A[ROW+5,ROW+2+J]=SHJ*((DCN-DDN+DAN)*SNJ+(DEN+DBN)*CSJ) ;
A[ROW+5,ROW+4+J]=CHJ*((DCN+DDN-DAN)*CSJ-(DEN+DBN)*SNJ) ;
A[ROW+5,ROW+6+J]=CHJ*((DCN+DDN-DAN)*SNJ+(DEN+DBN)*CSJ) ;
DAN=NY[J]+HY*NPI*NXY[J] ; DBN=NY[K]+HY*NXY[K]*NPI ;
A[ROW+6,COL+J]=SHB*(DAN*CSB+DBN*SNB);
A[ROW+6,COL+2+J]=SHB*(DAN*SNB-DBN*CSB) ;
DAN=NY[J]-HY*NXY[J]*NPI ; DBN=NY[K]-HY*NXY[K]*NPI ;
A[ROW+6,COL+4+J]=CHB*(DAN*CSB-DBN*SNB) ;
A[ROW+6,COL+6+J]=CHB*(DAN*SNB+DBN*CSB) ;
DAN=D2*IZB*NPI**4-2*A2B*(NPI*PE)**2*HZ ;
DCN=IZB*NPI**4-2*A2B*(PE*NPI)**2 ;
DDN=DCN*AV[J]+DAN*RJ ; DEN=DCN*AV[K]+DAN*RK ;
DAN=R2*IZB*NPI**4-NY[J]-HY*NPI*NXY[J] ; DBN=NY[K]+HY*NPI*NXY[K] ;
A[ROW+6,ROW+J]=SHJ*((DAN-DDN)*CSJ-(DBN+DEN)*SNJ) ;
A[ROW+6,ROW+2+J]=SHJ*((DAN-DDN)*SNJ+(DBN+DEN)*CSJ) ;
DAN=R2*IZB*NPI**4-NY[J]+HY*NPI*NXY[J] ; DBN=NY[K]-HY*NPI*NXY[K] ;
A[ROW+6,ROW+4+J]=CHJ*((DAN+DDN)*CSJ-(DEN-DBN)*SNJ) ;
A[ROW+6,ROW+6+J]=CHJ*((DAN+DDN)*SNJ+(DEN-DBN)*CSJ) ;
DCN=GY*VY[J]-R3*WN*NPI*NXY[J] ; DDN=GY*VY[K]+R3*WN*NPI*NXY[K] ;
DAN=MY[J]-GZB*NY[J] ; DBN=MY[K]-GZB*NY[K] ;
A[ROW+7,COL+J]=SHB*((DAN-DCN)*CSB+(DBN+DDN)*SNB) ;
A[ROW+7,COL+2+J]=SHB*((DAN-DCN)*SNB-(DBN+DDN)*CSB) ;
A[ROW+7,COL+4+J]=CHB*((DAN+DCN)*CSB+(DDN-DBN)*SNB) ;
A[ROW+7,COL+6+J]=CHB*((DAN+DCN)*SNB+(DBN-DDN)*CSB) ;
DCN=C1B*NPI**4+(CB-IOS*2*PE*PE)*NPI**2 ;
DDN=MY[J]-GZB*NY[J]+2*A1B*Y0*(PE*NPI)**2 ; DEN=MY[K]-GZB*NY[K] ;
DAN=DCN*RJ-2*A1B*Z0*(PE*NPI)**2*AV[J]+GY*VY[J]-R3*WN*NPI*NXY[J] ;
DBN=DCN*RK-2*A1B*Z0*(PE*NPI)**2*AV[K]-GY*VY[K]-R3*WN*NPI*NXY[K] ;
A[ROW+7,ROW+J]=SHJ*((DAN-DDN)*CSJ+(DBN-DEN)*SNJ) ;
A[ROW+7,ROW+2+J]=SHJ*((DAN-DDN)*SNJ-(DBN-DEN)*CSJ) ;
A[ROW+7,ROW+4+J]=CHJ*(-(DAN+DDN)*CSJ+(DBN+DEN)*SNJ) ;
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 A[ROW+7,ROW+6+J]=CHJ*(-(DAN+DDN)*SNJ-(DBN+DEN)*CSJ) ;
 A[ROW+8,COL+J]=-SHB*(NXY[J]*CSB+NXY[K]*SNB) ;
 A[ROW+8,COL+2+J]=-SHB*(NXY[J]*SNB-NXY[K]*CSB) ;
 A[ROW+8,COL+4+J]=CHB*(NXY[J]*CSB-NXY[K]*SNB) ;
 A[ROW+8,COL+6+J]=CHB*(NXY[J]*SNB+NXY[K]*CSB) ;
 DAN=NXY[J]+A2B*NPI**2*(AU[J]+NPI*(HZ-HY*AV[J]-D3*RJ)) ;
 DBN=NXY[K]+NPI**2*A2B*(AU[K]-NPI*(HY*AV[K]+D3*RK)) ;
 A[ROW+8,ROW+J]=SHJ*(DAN*CSJ+DBN*SNJ) ;
 A[ROW+8,ROW+2+J]=SHJ*(DAN*SNJ-DBN*CSJ) ;
 DAN=A2B*NPI**2*(AU[J]+NPI*(HZ+HY*AV[J]+D3*RJ))-NXY[J] ;
 DBN=A2B*NPI**2*(AU[K]+NPI*(HY*AV[K]+D3*RK))-NXY[K] ;
 A[ROW+8,ROW+4+J]=CHJ*(DAN*CSJ-DBN*SNJ) ;
 A[ROW+8,ROW+6+J]=CHJ*(DAN*SNJ+DBN*CSJ) ;
END END END ELSE BEGIN
FOR J=(1,1,J1) DO BEGIN  RJ=R[J] ;
AU[J]=(RJ**6/(NPI**3)+(NU-2)*RJ**4/NPI+NPI*(1-2*NU)*RJ**2+NPI**3*NU)/
(Z0**2)-2*LA*LA*(RJ**2/NPI+NU*NPI) ;AV[J]=-RJ*((RJ**6/(NPI**4)-(4+NU)
*RJ**4/(NPI**2)+(5+2*NU)*RJ**2-(2+NU)*NPI**2)/(Z0**2)-2*LA*LA*((RJ/
NPI)**2-2-NU)); VY[J]=NPI**2*RJ*(2-NU)-RJ**3 ; MY[J]=NU*NPI**2-RJ**2;
NXY[J]=(1-NU)*(AU[J]*RJ+NPI*AV[J])/2;MY[J]=AV[J]*RJ-1-NU*NPI*AU[J]
 END; IF J1 NEQ 4 THEN FOR J=(J1+1,1,4) DO BEGIN RJ=R[J] ;
AU[J]=(-RJ**6/(NPI**3)+(NU-2)*RJ**4/NPI-NPI*(1-2*NU)*RJ**2+NU*NPI**3)
/(Z0**2)+2*LA*LA*(RJ**2/NPI-NU*NPI) ; AV[J]=RJ*((RJ**6/(NPI**4)+(4+NU)
*RJ**4/(NPI**2)+(5+2*NU)*RJ**2+NPI**2*(2+NU))/(Z0*Z0)-2*LA*LA*((RJ/
NPI)**2+2+NU)); VY[J]=(2-NU)*RJ*NPI**2+RJ**3 ; MY[J]=NPI**2*NU+RJ**2;
NXY[J]=(1-NU)*(AU[J]*RJ+NPI*AV[J])/2;MY[J]=AV[J]*RJ+1+NU*NPI*AU[J] ;
END ;
FOR I=(1,1,B1) DO BEGIN  ROW=(I-1)*8 ;
IF I EQL 1 THEN COL=(BI-1)*8 ELSE COL=(I-2)*8 ;
FOR J=(1,1,J1) DO BEGIN  K=J+J1 ; RJ=R[J] ;
SHJ=CHB=1 ;    CHJ=SHB=EXP(-RJ*DY) ;
A[ROW+1,ROW+J]=SHJ ; A[ROW+1,ROW+K]=CHJ ;
A[ROW+1,COL+J]=-SHB ; A[ROW+1,COL+K]=-CHB ;
A[ROW+2,ROW+J]=AU[J]*SHJ ; A[ROW+2,ROW+K]=AU[J]*CHJ ;
A[ROW+2,COL+J]=-AU[J]*SHB ; A[ROW+2,COL+K]=-AU[J]*CHB ;
A[ROW+3,ROW+J]=-AV[J]*SHJ ; A[ROW+3,ROW+K]=AV[J]*CHJ ;
A[ROW+3,COL+J]=AV[J]*SHB ; A[ROW+3,COL+K]=-AV[J]*CHB ;
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A[ROW+4,ROW+J]=-RJ*SHJ ; A[ROW+4,ROW+K]=RJ*CHJ ;
A[ROW+4,COL+J]=RJ*SHB ; A[ROW+4,COL+K]=-RJ*CHB ;
DAN=VY[J]-HZB*NPI*NXY[J]; DBN=IYB*NPI**4-A1B*PE2*NPI**2 ;
DCN=(R1*AV[J]+D1*RJ)*IYB*NPI**4+A1B*PE2*RJ*HY*NPI**2-DAN ;
A[ROW+5,COL+J]=-SHB*DAN ; A[ROW+5,COL+K]=DAN*CHB ;
A[ROW+5,ROW+J]=(DBN-DCN)*SHJ ; A[ROW+5,ROW+K]=(DBN+DCN)*CHJ ;
DCN=HY*NPI*NXY[J] ; DAN=R2*IZB*NPI**4-NY[J] ;
DBN=(IZB*NPI**4-A2B*PE2*NPI**2)*AV[J]+(D2*IZB*NPI**4-A2B*PE2*HZ*NPI**
2)*RJ+DCN ;
A[ROW+6,COL+J]=(NY[J]+DCN)*SHB;A[ROW+6,COL+K]=(NY[J]-DCN)*CHJ ;
A[ROW+6,ROW+J]=(DAN-DBN)*SHJ ; A[ROW+6,ROW+K]=(DAN+DBN)*CHJ ;
DAN=MY[J]-GZB*NY[J] ; DBN=GY*VY[J]-R3*WN*NPI*NXY[J] ;
DCN=DAN+A1B*Y0*PE2*NPI**2 ;
DDN=DBN+(C1B*NPI**4-(CB+IOS*PE2)*NPI**2)*RJ-A1B*Z0*PE2*AV[J]*NPI**2;
A[ROW+7,COL+J]=(DAN-DBN)*SHB ; A[ROW+7,COL+K]=(DAN+DBN)*CHB ;
A[ROW+7,ROW+J]=(DDN-DCN)*SHJ ; A[ROW+7,ROW+K]=-(DDN+DCN)*CHJ ;
DAN=NXY[J] ; DBN=A2B*NPI**2*(AU[J]+NPI*(HZ-HY*AV[J]-D3*RJ)) ;
A[ROW+8,COL+J]=-DAN*SHB ; A[ROW+8,COL+K]=DAN*CHB ;
A[ROW+8,ROW+J]=(DBN+DAN)*SHJ ; A[ROW+8,ROW+K]=(DBN-DAN)*CHJ  END ;
FOR J=(J1+1,1,4) DO BEGIN K=J+J1 ; S=K+J2 ; RJ=R[J] ;
DAN=RJ*(I-1)*DY ; CSJ=COS(DAN) ; SNJ=SIN(DAN) ;IF I GEQ 2 THEN BEGIN
DAN=RJ*(I-1)*DY ; CSB=COS(DAN); SNB=SIN(DAN) END ELSE BEGIN
DAN=RJ*BI*DY ; CSB=COS(DAN) ; SNB=SIN(DAN) END ;
A[ROW+1,ROW+K]= CSJ ; A[ROW+1,ROW+S]= SNJ ;
A[ROW+1,COL+K]=-CSB ;  A[ROW+1,COL+S]=-SNB ;
A[ROW+2,ROW+K]= AU[J]*CSJ ; A[ROW+2,ROW+S]= AU[J]*SNJ ;
A[ROW+2,COL+K]=-AU[J]*CSB ;  A[ROW+2,COL+S]=-AU[J]*SNB ;
A[ROW+3,ROW+K]=-AV[J]*SNJ ;  A[ROW+3,ROW+S]= AV[J]*CSJ ;
A[ROW+3,COL+K]= AV[J]*SNB ; A[ROW+3,COL+S]=-AV[J]*CSB ;
A[ROW+4,ROW+K]=-RJ*SNJ ;      A[ROW+4,ROW+S]= RJ*CSJ ;
A[ROW+4,COL+K]= RJ*SNB ;     A[ROW+4,COL+S]=-RJ*CSB ;
DAN=VY[J]-HZB*NPI*NXY[J] ; DBN=IYB*NPI**4-A1B*PE2*NPI**2 ;
DCN=(R1*AV[J]+D1*RJ)*IYB*NPI**4+A1B*PE2*RJ*HY*NPI**2-DAN ;
A[ROW+5,COL+K]=-DAN*SNB ; A[ROW+5,COL+S]=DAN*CSB ;
A[ROW+5,ROW+K]=DBN*CSJ-DCN*SNJ ; A[ROW+5,ROW+S]=DBN*SNJ+DCN*CSJ ;
DCN=HY*NPI*NXY[J] ; DAN=R2*IZB*NPI**4+NY[J] ; DBN=(IZB*NPI**4-A2B*PE2
*NPI**2)*AV[J]+(D2*IZB*NPI**4-A2B*PE2*HZ*NPI**2)*RJ+DCN ;
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    A[ROW+6,COL+K]=-NY[J]*CSB+DCN*SNB; A[ROW+6,COL+S]=-NY[J]*SNB+DCN*CSB;
    A[ROW+6,ROW+K]=DAN*CSJ-DBN*SNJ : A[ROW+6,ROW+S]=DAN*SNJ+DBN*CSJ ;
    DAN=MY[J]+GZB*NY[J] ; DBN=GY*VY[J]-R3*WN*NP[*NXY[J] ;
    DCN=DAN+A1B*YO*PE2*NP[**2 ;
    DDN=DBN+(C1B*NP[**4+(C3-IOS*PE2)*NP[**2)*RJ-A1B*Z)*PE2*AV[J]*NP[**2 ;
    A[ROW+7,COL+K]=DAN*CSB-DBN*SNB : A[ROW+7,COL+S]=DAN*SNB+DBN*CSB ;
    A[ROW+7,ROW+K]=-DCN*CSJ+DDN*SNJ ; A[ROW+7,ROW+S]=-DCN*SNJ-DDN*CSJ ;
    DAN=NXY[J] ; DBN=A2B*NP[**2*(AU[J]+H/*JP[) ;
    DCN=A2B*NP[**3*(GY*AV[J]+D3*RJ)-NXY[J] ;
    A[ROW+8,COL+K]=-DAN*SNB ; A[ROW+8,COL+S]=DAN*CSB ;
    A[ROW+8,ROW+K]=DBN*CSJ-DCN*SNJ ; A[ROW+8,ROW+S]=DBN*SNJ+DCN*CSJ ;
    END END END ;
     IF COD EQL  1 THEN GO TO L3 ;
     DETERMINT(NC,A,&-18,DET[KI]) ;
     WRITE(LA,DLA,DET[KI],FT5) ;
     IF KI EQL 1 THEN GO TO L2  ;
     IF ABS(DLA/LA) LSS 0.00001 THEN IF OPT EQL 1 THEN BEGIN
     WRITE(N,LA,FM15) ; IF N LSS BN THEN GO TO L50 ELSE GO TO EXIT END
     ELSE BEGIN Q[NM]=LA ;  COD=1 ;
     FOR K=(1,1,NC) DO FOR L=(1,1,NC) DO A[K,L]=0 ; GO TO L6 END ;
     IF SIGN(DET[KI]) EQL SIGN(DET[KI-1]) THEN GO TO L2 ELSE BEGIN
     LA=LA-DLA ; DLA=DLA/2 ; DET[KI]=DET[KI-1] ; GO TO L2 END ;
     FOR I=1,2,3,4 DO RP[NM,I]=R[I] ; JC[NM]=J1 ;
L3:  FOR I=(1,1,NC) DO FOR J=(1,1,NC) DO A[I,J]=A[I,J]/4 ;
     FOR I=(1,1,NC1) DO C[I]=-A[I,NC] ;
     RSW=FALSE;   COJ=0 ;
     SOLVE(NC1,A, C,RSW,&-04,5 ,&-19,X,F1,F2) ; COJ=1 ;
F1:  IF COJ EQL 0 THEN BEGIN WRITE(FM11); GO TO L1 END ;
F2:  IF COJ EQL 0 THEN WRITE(FM12) ;
     FOR I=(1,1,BI) DO FOR J=(1,1,8) DO BEGIN  JI=(I-1)*8+J ;
    IF JI EQL 8*BI THEN AB[NM,J,BI]=1 ELSE
    AB[NM,J,I]=X[JI  ] END ;
    END ;
L50: END ;
    WRITE(FM7) ;
    FOR N=(1,1,BN) DO FOR M=(1,1,BM) DO BEGIN NM=(N-1)*BM+M ;
    FOR I=(1,1,BI) DO WRITE(N,M,I,FOR J=(1,1,8) DO AB[NM,J,I],FM8) END;
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    FOR N=(1,1,ND) DO BEGIN SUM=0 ; FOR J=(1,1,8) DO FOR I=(1,1,BI) DO
    SUM=SUM+AB[N,J,I]**2 ;   DAN=SQRT(SUM) ;
    FOR J=(1,1,8) DO FOR I=(1,1,BI) DO AB[N,J,I]=AB[N,J,I]/DAN : END ;
COMMENT  START TO CALCULATE PRE-BUCKLING DEFORMATION ;
    FOR N2=(1,1,BIGN) DO BEGIN N=2*N2-1 ; NPI=N*PI/LL ;
    CNM=NPI**4/(ZO**2)-PE2*NPI**2+1 ; SUM=0 ;
    FOR K=(1,1,BIGM) DO BEGIN KPI=2*K*PI/DY ;
    DKL=((2+NU)*NPI**2*KPI+KPI**3)/((NPI**2+KPI**2)**2)  ;
    SUM=SUM+2/((NPI**2+KPI**2)**2/(ZO**2)-PE2*NPI**2+1+DKL*KPI) END ;
    DAN=1/(DY*CNM)+SUM+ZO**2/(TYI*NPI**4-PE2*A13*NPI**2) ;
    DKL=-4*PE2*NU/(N*PI*CNM*DAN*LL) ;
    AC[N2,0]=-(DKL/DY+4*NU*PE2/(N*PI*LL))/CNM ;
    FOR M=(1,1,BIGM) DO BEGIN MPI=2*M*PI/DY :
    DBN=((2+NU)*NPI**2*MPI+MPI**3)/(NPI**2+MPI**2)**2 ;
    DAN=(MPI**2+NPI**2)**2/(ZO**2)-PE2*NPI**2+1+MPI*DBN ;
    AC[N2,M]=-2*DKL*COS(M*PI)/(DY*DAN) END ;
    END ;
    FOR N=(1,1,BIGN) DO FOR M=(0,1,BIGM) DO WRITE(N,M,AC[N,M],FM9) ;
COMMENT  STABILITY DETERMINANT ;
L8: FOR K=(1,1,BN) DO FOR L=(1,1,BM) DO BEGIN KL=(K-1)*BM+L;KPI=K*PI/LL;
    KPI=TN*PI/LL ;
    FOR I=(1,1,4) DO RKL[I]=RR[KL,I] ; JCKL=JC[KL] ;
    FOR N=(1,1,BN) DO FOR M=(1,1,BM) DO BEGIN NM=(N-1)*BM+M;NPI=N*PI/LL ;
    NPI=TN*PI/LL ;
    FOR I=(1,1,4) DO RNM[I]=RR[NM,I] :  JCNM=JC[NM] ;
    SUM=0 ;
    FOR NP2=(1,1,BIGN) DO FOR MP=(0,1,BIGM) DO BEGIN NP=2*NP2-1 ;
    MPP=2*MP*PI/DY ; SUM1=SUM2=SUM3=0 ;
    NPP=NP*PI/LL ;
    FOR I=(1,1,BI) DO BEGIN YI=(I-1)*DY;
    FOR J=(1,1,8) DO BEGIN DA1[J]=AB[NM,J,I] ;
    DA3[J]=AB[KL,J,I] ; DA4[J]=AB[KM,J,I] END ;
    FOR S=(0,1,BS) DO BEGIN YT=H*S ; YY=YI+YT ;
    DAN=F(0,JCKL,RKL,DA3,YY) ; DBN=F(1,JCNM,RNM,DA1,YY) ;
    Y1[S]=DAN*DBN*GG(1,MP,DY,YT) ; DBN=F(0,JCNM,RNM,DA1,YY) ;
    Y2[S]=DAN*DBN*GG(2,MP,DY,YT) ; DBN=F(2,JCNM,RNM,DA1,YY) ;
    Y3[S]=DAN*DBN*GG(0,MP,DY,YT) END : SUM1=SUM1+SIMP(BS,H,Y1) ;
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    SUM2=SUM2+SIMP(BS,H,Y2) ; SUM3=SUM3+SIMP(BS,H,Y3) END ;
    INTSIN(TN,NP,TN,LL,ISN,ICS) ;
    SUM=SUM+AC[NP2,MP]*(ISN*(((NPI*NPP)**2+NPI**4)*SUM2+(NPP**4+
    (NPI*NPP)**2)*SUM3-4*(NPI*NPP)**2*SUM1)+ICS*2*((NPI*NPP**3+
    NPI**3*NPP)*SUM1-NPI**3*NPP*SUM2-NPI*NPP**3*SUM3))
    END ; A[KL,NM]=SUM ; END ;
L9: FOR M=(1,1,BM) DO BEGIN KM=(K-1)*BM+M ; SUM6=0 ; JCKM=JC[KM] ;
    FOR I=(1,1,4) DO RKM[I]=RR[KM,I] ;
    FOR I=(1,1,BI) DO BEGIN YI=(I-1)*DY ;
    FOR J=(1,1,8) DO DA4[J]=AB[KM,J,I] ;
    FOR J=(1,1,8) DO DA3[J]=AB[KL,J,I] ;
    FOR S=(0,1,BS) DO BEGIN YT=S*H ; YY=YI+YT; DAN=F(0,JCKL,RKL,DA3,YY);
    DBN=KPI**2*(2*Q[KM]**2-PE2)*(F(4,JCKM,RKM,DA4,YY)-2*KPI**2*
    F(2,JCKM,RKM,DA4,YY)+KPI**4*F(0,JCKM,RKM,DA4,YY)) ;
    Y[S]=DAN*DBN END ;
    SUM6=SUM6+SIMP(BS,H,Y) ;    END ;
    A[KL,KM]=A[KL,KM]+SUM6*LL/2 END END ;
    GO TO L15 ;
    DETERMINT (ND,A,&-28,DTR[KK]) ;    WRITE(PF,DP,DTR[KK],FM5) ;
L15: DTR[KK]=A[1,1] ; WRITE(PF,DP,DTR[KK],FM5) ;
    IF KK EQL 1 THEN GO TO L1 ;
    IF SIGN(DTR[KK]) EQL SIGN(DTR[KK-1]) THEN GO TO L1 ELSE BEGIN
    PE=PE-DP ; DP=DP/2 ; DTR[KK]=DTR[KK-1] ; GO TO L1 END ;
EXIT: END ;
    END ;
```

REFERENCES

1. Abraham, L. H. and Lowy, M. J., "Shell Instability Problems As Related to Design," "Collected Papers on Instability of Shell Structures-1962," NASA TN D-1510, 1962, pp. 1-9.

2. Hedgepeth, J. M. and Hall, D. B., "Stability of Stiffened Cylinders," AIAA Journal, Vol. 3, No. 12, December, 1965, pp. 2275-2286.

3. Bress, M., Cours de Mechanique Applique, Paris, France, 1859, pp. 323-338.

4. Bryan, G. H., "Application of the Energy Test to the Collapse of a Long Thin Pipe Under External Pressure," Proceeding of the Cambridge Philosophical Society, Cambridge, England, Vol. 6, 1888, pp. 287-292.

5. Southwell, R. V., "On the Collapse of Tubes by External Pressure," Philosophical Magazine, Vol. 25, Part 1, May, 1913, pp. 687-698, Vol. 26, Part 2, September, 1913, pp. 502-511, Vol. 29, Part 3, January, 1915, pp. 67-77.

6. Flügge, W., "Die Stabilitat der Kreiszylin derschale," Ingenieur-Archiv, December, 1932, pp. 463-506.

7. Dschou, Dji-Djüän, "Die Druckfestigkett Versteifter Zylindrischer Schallen," Luftfahrt-forschung, Vol. 11, February 6, 1935, pp. 223-234.

8. Taylor, J. L., "The Stability of Monocoque in Compression," Great Britain Aeronautical Research Council Reports and Memoranda, No. 1679, 1935.

9. Timoshenko, Stephen, Theory of Elastic Stability, McGraw-Hill Book Co., Inc., New York, N. Y., 1936.

10. Nissen, O., "Knickversuche mit Versteiften Wellblechsehalen bei reiner Druckbeanspruchung," Jahrb. 1937 der deutschen Luftfahrtforschung, pp. I 452-I 458.

11. Rydner, E. I., "General Instability of Semimonocoque Cylinders," Air Commerce Bulletin, Department of Commerce, Vol. 9, 1938, pp. 241-246.

12. Donnell, L. H., "Stability of Thin Walled Tubes Under Torsion," NACA Report, No. 479, 1933.

13. Hoff, N. J., "General Instability of Monocoque Structures in Pure Bending," Journal of Royal Aeronautic Society, Vol. XLII, No. 328, April, 1938, pp. 291-346.

14. Wang, T. K., "Torsion and Shear Effect of Members Upon General Instability of Semimonoque Structure Under the Compression," Journal of Applied Mechanics, Vol. 14, No. 3, September, 1947, pp. A 177- A 182.

15. Salerno, V. L. and Levine, B., "General Instability of Reinforced Shell Under Hydrostatic Pressure," Polytechnic Institute of Brooklyn, Report No. 189, September, 1951.

16. Kendricks, S., "The Buckling Under External Pressure of Circular Cylindrical Shells with Evenly Spaced, Equal Strength, Circular Ring Frame, Part I, II and III," Great Britain Naval Construction Research Establishment, Reports No. 211, 243 and 244, 1955.

17. Nash, W. A., "Buckling of Multiple-Buy Ring-Reinforced Cylindrical Shell Subject to Hydrostatic Pressure," Journal of Applied Mechanics, Vol. 20, No. 4, December, 1953.

18. Stein, M., Sanders, S. L., and Crate, H., "Critical Stress in Torsion," NACA Report, No. 989, 1951.

19. Anderson, M. S. and Card, M. F., "Buckling of Ring-Stiffened Cylinder Under a Pure Moment and a Non-Uniform Temperature Distribution," NASA TN D-1513, November, 1962.

20. Wood, W. G., "Collapse Under End Load of Pressurized Axial Stiffened Thin Cylinders," Journal of Mechanical Engineering Science, Vol. 7, No. 4, December, 1965, pp. 469-481.

21. Mckenzie, K. I., "The Buckling of Pressurized Stiffened Cylinder Under Axial Load," Great Britain Aeronautical Research Council Reports and Memoranda.

22. Hoff, N. J., Boley, B. A. and Klein, B., "Stress in and General Instability of Monocoque Cylinder with Cutouts."

Part III	NACA	TN	1263	May 1947
IV	NACA	TN	1264	February 1948
V	NACA	TN	1435	January 1948
VI	NACA	TN	1436	March 1948
VII	NACA	TN	1962	October 1949
VIII	NACA	TN	1963	October 1949

23. Dickson, J. N. and Brolliar, R. H., "The General Instability of Eccentrically Stiffened Cylindrical Shells Under Axial Compression and Lateral Pressure," NASA CR 1280, January, 1969.

24. Simitses, G. J., "Instability of Orthotropic Cylindrical Shells Under Combined Torsion and Hydrostatic Pressure," AIAA Journal, Vol. 5, No. 8, August, 1967.

25. Tokugawa, V. Y., "Model Experiment on the Elastic Stability of Closed and Crossed-Stiffened Circular Cylinder Under Uniform External Pressure," Society of Naval Architects W.E.C., Part 2, No. 651, 1929.

26. Galletly, G. D., Slankard, R. C. and Wenk, E., "General Instability of Ring-Stiffened Cylindrical Shells Subject to External Hydrostatic Pressure - A Compression of Theory and Experimental," Journal of Applied Mechanics, Vol. 25, No. 2, June, 1958.

27. Harris, L. A., Suer, H. S. and Skene, W. T., "Model Investigation of Unstiffened and Stiffened Circular Shell," Experimental Mechanics, I. 7, 1-9, July, 1961.

28. Peterson, J. P., Whitley, R. O. and Deaton, J. W., "Structural Behavior and Compressive Strength of Circular Cylinders with Longitudinal Stiffening," NASA TN D-1251, May, 1962.

29. Shang, J. C., Marulic, W. J. and Sturn, R. G., "Buckling of Longitudinally Stiffened Cylinder," American Society of Civil Engineering Proceedings, Vol. 90 (J. Str. Div.) n ST8, October, 1964.

30. Milligan, R., Gerard, G., Lakshmikantham, C. and Becker, H., "General Instability of Orthotropic Stiffened Cylinder Under Axial Compression," AIAA Journal, Vol. 4, No. 11, November, 1966.

31. Singer, J., "Influence of Stiffener Geometry and Spacing on the Buckling of Axially Compressed Cylindrical and Conical Shells," Technion-Israel Institute of Technology, Department of Aeronautical Engineering, TAE, Report 68, October, 1967.

32. Tsao, C. H. and Ching, A., "Test of, and Semiempirical Formulas for, Instability of Longitudinally Stiffened Shells," Journal of Spacecraft, Vol. 5, No. 2, February, 1968.

33. Van der Neut, A., "The General Instability of Stiffened Cylindrical Shells under Axial Compression," Report S 314, National Aeronautics Research Institute, (Amsterdam), 1947.

34. Baruch, M. and Singer J., "Effect of Eccentricity of Stiffeners on General Instability of Stiffened Cylindrical Shell Under Hydrostatic Pressure," Journal of Mechanical Engineering Science, Vol. 5, pp. 23-27, 1963.

35. Singer, J., Baruch, M. and Harari, O., "On Stability of Eccentrically Stiffened Cylindrical Shell Under Axial Compression," Haifa, Israel Institute of Technology, Department of Aeronautical Engineering, TAE, Report No. 44, December, 1965.

36. Singer, J., Baruch, M. and Harari, O., "Further Remarks on the Effect of Eccentricity of Stiffeners on the General Instability of Stiffened Cylindrical Shells," Journal of Mechanical Engineering Science, Vo. 8, p. 363, 1966.

37. Block, D. L., Card, M. F. and Mikulas, M. M., "Buckling of Eccentrically Stiffened Orthotropic Cylinders," NASA TN D-2960, 1965.

38. McElman, J. A., Mikulas, M. M. and Stein, M., "Static and Dynamic Effects of Eccentric Stiffening of Plates and Cylindrical Shell," Presented at the 2nd AIAA Meeting and Technical Display, San Francisco, July, 1965, AIAA Journal, Vol. 4, No. 5, May, 1966, pp. 887-894.

39. Simitses, G. J., "A Note on the General Instability of Eccentrically Stiffened Cylinders," Journal of Aircraft, September - October, 1967, pp. 473-475; "Buckling of Eccentrically Stiffened Cylinders Under Torsion," AIAA Journal, Vol. 6, No. 10, October, 1968, pp. 1856-1860; and "Buckling of Eccentrically Stiffened Cylinders Under Combined Loads," AIAA Journal, Vol. 7, No. 2, February, 1969, pp. 335-337.

40. Houghton, D. S. and Chan, A. S. L., "Design of a Pressurised Missile Body," Aircraft Engineering, Vol. 32, p. 330, 1960.

41. Card, M. F., "Preliminary Results of Compression Tests on Cylinders with Eccentric Longitudinal Stiffeners," NASA TMX-10004, September, 1964.

42. DeLuzio, A. J., Stuhlman, C. E. and Almorth, B. O., "Influence of Stiffeners Eccentricity and End Moment on the Stability of Cylinders in Compression," Presented at AIAA 6th Structures and Materials Conf. Palm Springs, April, 1965, AIAA Journal, Vol. 4, No. 5, May, 1966, pp. 872-877.

43. Garkisch, H. D., Geier, B. and Seggelke, P., "Beulversuche an Langsversteiften Zylinderschalen," Deutshe Luft-und Raumfahrt, Forschungsbericht.

44. Thielemann, W. and Esslinger, M., "Uber den Einfluss der Exzentrizitat von Langssteifen auf die Axile Beullast Dunnwandiger Kreiszylinerschalen," Stahlbau 34, 332, 1965.

45. Brush, D. O., "Initial Postbuckling Behavior of Stiffened Cylindrical Shell," Lockheed Missiles and Space Co., Report LMSC 6-77-67-52, November, 1967 (Microfilm N68-25992).

46. Hutchinson, J. W. and Amaaigo, J. C., "Imperfection-Sensitivity of Eccentrically Stiffened Cylindrical Shells," AIAA Journal, Vol. 5, No. 3, pp. 392-401, March, 1967.

47. Koiter, W. T., "On the Stability of Elastic Equilibrium," Thesis, Delft, (H. J. Paris, Amsterdan, Holland, 1945), in Dutch.

48. Koiter, W. T., "Elastic Stability and Post-Buckling Behavior," Non-linear Problems, edited by R. E. Langer, (University of Wisconsin Press, Madison, Wisconsin, 1963).

49. Koiter, W. T., "General Equation of Elastic Stability for Thin Shells," Donnell Testimonia Meeting, Houston, Texas, April, 1966.

50. Jones, Robert M., "Buckling of Circular Shells with Multiple Orthotropic Layers and Eccentric Stiffeners," Aerospace Corporation, Report TR-0158 (53820-10)-1 (Microfilm N68-12095).

51. Moe, J., "Stability of Rib-Reinforced Cylindrical Shells Under Lateral Pressure," Memories Abhandlungen Publications, Vol. 18, 1958, International Association for Bridge and Structural Engineering.

52. Block, D. L., "Influence of Ring Stiffeners on Instability of Orthotropic Cylinders in Axial Compression," NASA TN D-2482, October, 1964.

53. Van der Neut, "General Instability of Orthogonal Stiffened Cylindrical Shell," "Collected Papers on Instability of Shell Structures - 1962," NASA TN D-1510, 1962, pp. 309-319.

54. MacNeal, K. H., Winemiller, A. F. and Bailie, J. A., "Elastic Stability of Cylindrical Shell Reinforced by One or Two Frames and Subjected to External Radial Pressure," AIAA Journal, Vol. 4, No. 8, pp. 1431-1433, August, 1966.

55. Ross, C. T. F., "Collapse of Ring-Reinforced Circular Cylinder," Engineer, Vol. 221, January 28, 1966, pp. 162-163.

56. Singer, J., Haftka, R., "Buckling of Discretely Ring Stiffened Cylindrical Shell," Haifa - Israel Institute of Technology, Department of Aeronautical Engineering, TAE Report 67, August, 1967.

57. Wang, J. T. S., "Orthogonally Stiffened Cylindrical Shell Subjected to Internal Pressure," Sixth Congress of International Council of Aeronautical Science, September, 1968. AIAA Journal, March, 1970.

58. Egle, D. M., and Sewall, J. L., "An Analysis of Free Vibration of Orthogonally Stiffened Cylindrical Shells with Stiffeners Treated as Discrete Elements," AIAA Journal, Vol. 6, No. 6, March, 1968.

59. Rinehart, S. A., "Vibration of a Longitudinally-Stiffened Cylindrical Shell," Dissertation, Georgia Institute of Technology, 1970.

60. McDonald, D., "A Problem in the Free Vibration of Stiffened Cylindrical Shell," AIAA Journal, Vol. 8, No. 2, February, 1970.

61. Fung, Y. C. and Sechler, E. E., "Instability of Thin Elastic Shells," Proceeding of the First Symposium on Naval Structural Mechanics, Edited by J. N. Goodier and N. J. Hoff, Pergamon Press, 1960.

62. Vlasov, V. Z., Thin Walled Elastic Beams, Translated from Russian by Y. Schechtman, The Israel Program for Scientific Translation, Washington, D. C., 1961.

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